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Characterization of Colombeau Generalized Functions by their Pointvalues

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Abstract. A new characterization of elements of Colombeau-type algebras of generalized functions is presented. Since Colombeau functions are not uniquely determined by their pointvalues as defined in [1], [2], a concept of generalized points suitable for securing such a pointvalue description is introduced. This characterization provides an affirmative answer to an open question in the theory of algebras of generalized functions and enables a direct transfer of methods from classical analysis to generalized functions.

1. Introduction

In the sequential approach to the nonlinear theory of generalized functions, differential algebras are constructed as factor algebras

$$\mathcal{F}(\Omega) = \mathcal{A}(\Omega)/\mathcal{I}(\Omega)$$

where Ω is an open subset of $\mathbb{R}^n, \mathcal{A}(\Omega)$ is a differential subalgebra of $(\mathcal{C}^\infty(\Omega))^I$ for some infinite index set I and $\mathcal{I}(\Omega)$ is a differential ideal in $\mathcal{A}(\Omega)$. These algebras contain the space of distributions $\mathcal{D}'(\Omega)$ as a linear subspace and serve as a framework for solving nonlinear partial differential equations with singular data. Typical examples are the algebras of Colombeau [1, 2], Egorov [3], Rosinger [11, 12], as well as the ultrapower construction of the nonstandard space ${}^*\mathcal{C}^\infty(\Omega)$, see e.g. [5, 13]. For a general survey of these methods we refer to [10]. In the mentioned cases, the ring of generalized numbers $\mathcal{K}$ can be defined as $\mathcal{F}(\mathbb{R}^0)$ and is isomorphic to the ring of constants in the differential algebraic sense in case Ω is connected. Further, for $u \in \mathcal{F}(\Omega), x \in \Omega$, the pointvalue $u(x)$ can be defined as a generalized number in $\mathcal{K}$ by using representatives.

In this paper we address the question whether generalized functions $u \in \mathcal{F}(\Omega)$ are uniquely determined by their pointvalues. Example 2.1 below shows that this is not

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the case in general. It may serve as a counterexample not only in the Colombeau setting, but in the other algebras as well. Moreover, in Nonstandard Analysis it is well-known that an internal smooth function, an element of ${}^*\mathcal{C}^\infty(\Omega)$, is not uniquely determined by its pointvalues on the standard points $x \in \Omega$ (take again Examle 2.1 with ε infinitesimal), but is determined by its pointvalues on all points $x \in {}^*\Omega$ (the nonstandard points inclusive). This is actually an immediate consequence of the transfer principle.

It has been an open question (Problem 27.4 in [9]) whether a similar assertion is true for Colombeau algebras. The purpose of this paper is to give an affirmative answer. Guided by nonstandard principles, we introduce generalized points of Ω (as Ω-valued generalized maps on $\mathbb{R}^0$) and show that members of the Colombeau algebra $\mathcal{G}(\Omega)$ are determined by their values on compactly supported generalized points. We also show that elements of the algebra $\mathcal{G}_\tau(\Omega)$, playing a central role in the theory of Fourier transforms in the Colombeau setting, are determined by their pointvalues under suitable restrictions on the set Ω. The proofs in the Colombeau setting require rather intricate estimates with respect to growth in the regularization parameters. We also observe that an analogous pointvalue characterization is valid in the Egorov setting.

Finally, we would like to bring this concept in relation to the notion of a pointvalue of a distribution in the sense of Łojasiewicz [8]. Indeed, Łojasiewicz has shown that if a distribution $w \in \mathcal{D}'(\Omega)$ has the pointvalue zero at every point of Ω, then w vanishes as a distribution. However, the pointvalue of a distribution at an arbitrary point generally does not exist, in contrast to the situation in the setting of algebras of generalized functions.. In fact, if a distribution has a pointvalue in Łojasiewicz' sense at every point, then it is a function of first Baire class [8]. In case the Łojasiewicz pointvalue exists at some point $x \in \Omega$, it is associated with the generalized pointvalue in the sense of Colombeau (see [1, 2]).

In order to reduce notational complications we are going to work in the so-called simplified variants of the Colombeau algebras (see [1, 2, 9]). Thus, let Ω be an open subset of $\mathbb{R}^n$. Throughout this paper, for elements of the space $\mathcal{C}^\infty(\Omega)^I$ of sequences of smooth functions indexed by $\varepsilon \in I = (0, \infty)$ we shall use the notation $(u_\varepsilon)_\varepsilon$ (so $u_\varepsilon \in \mathcal{C}^\infty(\Omega)$ for $\varepsilon \in I$). We set $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$, where

$$\begin{aligned}
\mathcal{E}_M(\Omega) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^I : \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_o^n\ \exists p \in \mathbb{N} \text{ with} \\
&\qquad \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0\} \\
\mathcal{N}(\Omega) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(\Omega)^I : \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_o^n\ \forall q \in \mathbb{N} \\
&\qquad \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{q}) \text{ as } \varepsilon \to 0\}.
\end{aligned}$$

$\mathcal{G}(\Omega)$ is a differential algebra containing $\mathcal{D}'(\Omega)$ as a linear subspace and $\mathcal{C}^\infty(\Omega)$ as a faithful subalgebra, the embedding depending on a 'mollifier' from $\mathcal{S}(\mathbb{R}^n)$, the space of rapidly decreasing functions on $\mathbb{R}^n$ (in contrast to the 'full' version of the Colombeau algebra for which a canonical embedding exists). Equivalence classes of sequences $(u_\varepsilon)_\varepsilon$ will be denoted by $\mathrm{cl}[(u_\varepsilon)_\varepsilon]$. Second, we consider the algebra $\mathcal{G}_\tau(\Omega) = \mathcal{E}_\tau(\Omega)/\mathcal{N}_\tau(\Omega)$

of tempered generalized functions, where

$$\mathcal{O}_M(\Omega) = \{f \in \mathcal{C}^\infty(\Omega) : \forall \alpha \in \mathbb{N}_o^n \ \exists p > 0 \ \sup_{x\in\Omega}(1+|x|)^{-p}|\partial^\alpha f(x)| < \infty\}$$

$$\mathcal{E}_\tau(\Omega) = \{(u_\varepsilon)_\varepsilon \in (\mathcal{O}_M(\Omega))^I : \forall \alpha \in \mathbb{N}_o^n \ \exists p > 0 \ \sup_{x\in\Omega}(1+|x|)^{-p}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-p}) \ (\varepsilon \to 0)\}$$

$$\mathcal{N}_\tau(\Omega) = \{(u_\varepsilon)_\varepsilon \in (\mathcal{O}_M(\Omega))^I : \forall \alpha \in \mathbb{N}_o^n \ \exists p > 0 \ \forall \ q > 0 \ \sup_{x\in\Omega}(1+|x|)^{-p}|\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{q}) \ (\varepsilon \to 0)\}$$

Take $\rho \in \mathcal{S}(\mathbb{R}^n)$ with $\int \rho(x)\,dx = 1$ and $\int \rho(x)\,x^\alpha\,dx = 0$ for $|\alpha| \geq 1$. Then

$$\iota : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{G}_\tau(\mathbb{R}^n)$$
$$w \to \mathrm{cl}[(w * \rho_\varepsilon)_\varepsilon]$$

is a linear embedding commuting with partial derivatives and rendering

$$\mathcal{O}_C(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \exists p > 0 \ \forall \alpha \in \mathbb{N}_o^n \ \sup_{x\in\mathbb{R}^n}(1+|x|)^{-p}|\partial^\alpha f(x)| < \infty\}$$

a faithful subalgebra. The ring of constants of the above algebras, i.e. the ring of generalized numbers will be denoted by $\mathcal{K}$ if the smooth functions u_ε in the above definitions are supposed to be $\mathbb{K}$-valued (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A detailed exposition of the constructions outlined in this section can be found in [6]. We emphasize that all of the results to be presented in the sequel carry over to the respective full versions of the Colombeau algebras as well.

2. Pointvalue Characterization

As noted in the Introduction, the pointvalue of an element $U \in \mathcal{G}(\Omega)$ or $\mathcal{G}_\tau(\Omega)$ at $x \in \Omega$ is defined as the class of $(u_\varepsilon(x))_\varepsilon$, where $(u_\varepsilon)_\varepsilon$ is a representative of U. Here is the announced example that elements of $\mathcal{G}(\Omega)$ are not uniquely determined by prescribing their pointvalues:

Example 2.1. Take some $\varphi \geq 0 \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subseteq [-1,1]$ and $\int \varphi = 1$ and set $u_\varepsilon(x) = \varphi_\varepsilon(x - \varepsilon)$, where $\varphi_\varepsilon(y) := \frac{1}{\varepsilon}\varphi(\frac{y}{\varepsilon})$. Then $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R})$, so $U := \mathrm{cl}[(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R})$. It is easily seen that every pointvalue of every derivative of U is 0 in $\mathcal{K}$. But clearly $U \neq 0$ in $\mathcal{G}(\mathbb{R})$.

To remedy this situation we consider 'generalized points' in the following sense:

Definition 2.2. Let Ω be an open subset of $\mathbb{R}^n$. On

$$\Omega_M := \{(x_\varepsilon)_\varepsilon \in \Omega^I : \exists p > 0 \ \exists \ \eta > 0 \ |x_\varepsilon| \leq \varepsilon^{-p} \ (0 < \varepsilon < \eta)\}$$

we introduce an equivalence relation by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \ \Leftrightarrow \ \forall q > 0 \ \exists \eta > 0 \ |x_\varepsilon - y_\varepsilon| \leq \varepsilon^q \ (0 < \varepsilon < \eta)$$

and set $\widetilde{\Omega} := \Omega_M/\sim$. The set of *compactly supported points* is

$$\widetilde{\Omega}_c = \{\widetilde{x} \in \widetilde{\Omega} : \exists \text{ representative } (x_\varepsilon)_\varepsilon \ \exists K \subset\subset \Omega \ \exists \eta > 0 \ : \ x_\varepsilon \in K, \, \varepsilon \in (0, \eta)\}$$

It is clear that if the $\widetilde{\Omega}_c$-property holds for one representative of $\widetilde{x} \in \widetilde{\Omega}$ then it holds for every representative. Also, for $\Omega = \mathbb{K}$ we have $\widetilde{\mathbb{K}} = \mathcal{K}$. Our first observation is that generalized functions can be evaluated at generalized points:

Proposition 2.3. *Let $U \in \mathcal{G}(\Omega)$ (resp. $U \in \mathcal{G}_\tau(\Omega)$) and $\widetilde{x} \in \widetilde{\Omega}_c$ (resp. $\widetilde{x} \in \widetilde{\Omega}$). Then the pointvalue of U at $\widetilde{x} = \mathrm{cl}[(x_\varepsilon)_\varepsilon]$, $U(\widetilde{x}) := \mathrm{cl}[(u_\varepsilon(x_\varepsilon))_\varepsilon]$ is a well-defined element of $\mathcal{K}$.*

Proof. If $\widetilde{x} = \mathrm{cl}[(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c$, there exists some $K \subset\subset \Omega$ such that $x_\varepsilon \in K$ for ε small. Since $U \in \mathcal{G}(\Omega)$ it follows that $|u_\varepsilon(x_\varepsilon)| \le \sup_{x\in K} |u_\varepsilon(x)| \le \varepsilon^{-p}$ for small ε. Next we show that $\widetilde{x} \sim \widetilde{y}$ implies $U(\widetilde{x}) \sim U(\widetilde{y})$:

$$|u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)| \le |x_\varepsilon - y_\varepsilon| \int_0^1 |\nabla u_\varepsilon(x_\varepsilon + \sigma(y_\varepsilon - x_\varepsilon))| d\sigma \tag{2.1}$$

The claim now follows since $x_\varepsilon + \sigma(y_\varepsilon - x_\varepsilon)$ remains within some compact subset of Ω for small ε: The first factor is eventually smaller than any ε^q while the second is bounded by some ε^{-p}. Next, if $(w_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$ then $(w_\varepsilon(x_\varepsilon))_\varepsilon \sim 0$ again because x_ε stays within some compact set for ε small. If $(x_\varepsilon)_\varepsilon \in \Omega_M$ and $(u_\varepsilon)_\varepsilon \in \mathcal{E}_\tau(\Omega)$ then

$$|u_\varepsilon(x_\varepsilon)| \le \varepsilon^{-p}(1 + |x_\varepsilon|)^p \le \varepsilon^{-p}(1 + \varepsilon^{-p_1})^p \le \varepsilon^{-p'}$$

for small ε, so $(u_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{E}$. If $\widetilde{x} \sim \widetilde{y}$ then the right hand side of (2.1) is dominated by

$$|x_\varepsilon - y_\varepsilon|(1 + |x_\varepsilon| + |y_\varepsilon - x_\varepsilon|)^p \varepsilon^{-p} \le C\varepsilon^q (1 + \varepsilon^{-p_1})^p \varepsilon^{-p}$$

for arbitrary q and small ε, so $(u_\varepsilon(x_\varepsilon))_\varepsilon - (u_\varepsilon(y_\varepsilon))_\varepsilon \sim 0$. By similar arguments, $(w_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\Omega)$ implies $(w_\varepsilon(x_\varepsilon))_\varepsilon \sim 0$. □

As the main result of the paper, we show that contrary to usual pointvalues, pointvalues on generalized points characterize elements of $\mathcal{G}(\Omega)$:

Theorem 2.4. *If Ω is an open subset of $\mathbb{R}^n$ then*

$$U = 0 \ \textit{in} \ \mathcal{G}(\Omega) \ \Leftrightarrow \ U(\widetilde{x}) = 0 \ \textit{in} \ \mathcal{K} \ \textit{for all} \ \widetilde{x} \in \widetilde{\Omega}_c.$$

Proof. ⇒: follows directly from (the proof of) 2.3.
⇐: If $U \neq 0$ in $\mathcal{G}(\Omega)$ then

$$\exists K \subset\subset \Omega \ \exists \alpha \in \mathbb{N}_o^n \ \exists q > 0 \ \forall \eta > 0 \ \exists 0 < \varepsilon < \eta : \ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| > \varepsilon^q. \tag{2.2}$$

We choose α with the above property in such a way that $|\alpha|$ is minimal. (2.2) yields the existence of sequences $\varepsilon_k \to 0$ and $x_k \in K$ such that $|\partial^\alpha u_{\varepsilon_k}(x_k)| \ge \varepsilon_k^q$ for all

$k \in \mathbb{N}$. For $\varepsilon > 0$ we set $x_\varepsilon = x_k$ for $\varepsilon_{k+1} < \varepsilon \leq \varepsilon_k$, $k \in \mathbb{N}$. Then $(x_\varepsilon)_\varepsilon \in \Omega_M$ and has values in K, so $\widetilde{x} = \mathrm{cl}[(x_\varepsilon)_\varepsilon]$ belongs to $\widetilde{\Omega}_c$. Also, from the above we have $\partial^\alpha U(\widetilde{x}) \neq 0$ in $\mathcal{K}$. We have to distinguish two cases:

(i) $\alpha = 0$. Then $U(\widetilde{x}) \neq 0$ and the proof is completed.

(ii) $\alpha \neq 0$. We will show that this leads to a contradiction. Since $|\alpha|$ was assumed to be minimal, for any $\beta \in \mathbb{N}_o^n$, $|\beta| = |\alpha| - 1$ and $L \subset\subset \Omega$ we have

$$\forall r > 0\ \exists \eta > 0\ \forall 0 < \varepsilon < \eta: \ \sup_{x \in L} |\partial^\beta u_\varepsilon(x)| \leq \varepsilon^r \tag{2.3}$$

We may assume that $\alpha_1 \neq 0$. Set $\beta := (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n)$, $\gamma := (\alpha_1 + 1, \alpha_2, \ldots, \alpha_n)$ and $x = (x_1, x')$ for $x \in \mathbb{R}^n$. Since $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ it follows that

$$\exists p > 0\ \exists \eta > 0\ \forall 0 < \varepsilon < \eta: \ \sup_{x \in L} |\partial^\gamma u_\varepsilon(x)| \leq \varepsilon^{-p}$$

Choose $L \subset\subset \Omega$ such that $K \subseteq L^\circ$ (where L° denotes the interior of L). Then for k sufficiently large we have

$$\begin{aligned} |\partial^\alpha u_{\varepsilon_k}(y_1, x_{2,k}, \ldots, x_{n,k})| &= |\partial^\alpha u_{\varepsilon_k}(x_k) + \int_{x_{1,k}}^{y_1} \partial^\gamma u_{\varepsilon_k}(\xi, x'_k) d\xi| \geq \\ &\geq \varepsilon_k^q - \varepsilon_k^{-p} |y_1 - x_{1,k}| \geq \tfrac{1}{2}\varepsilon_k^q \end{aligned}$$

provided that $|y_1 - x_{1,k}| \leq \frac{1}{2}\varepsilon_k^{p+q}$ and that k is in addition so large that the line connecting x_k and $(y_1, x_{2,k}, \ldots, x_{n,k})$ is contained in L. Setting $\overline{x}_k := (x_{1,k} + \frac{1}{2}\varepsilon_k^{p+q}, x_{2,k}, \ldots, x_{n,k})$ we obtain

$$\begin{aligned} |\partial^\beta u_{\varepsilon_k}(\overline{x}_k)| &= |\partial^\beta u_{\varepsilon_k}(x_k) + \int_{x_{1,k}}^{\overline{x}_{1,k}} \partial^\alpha u_{\varepsilon_k}(\xi, x'_k) d\xi| \geq \\ &\geq -\varepsilon_k^r + |\overline{x}_{1,k} - x_{1,k}| \tfrac{1}{2}\varepsilon_k^q = -\varepsilon_k^r + \tfrac{1}{4}\varepsilon_k^{p+2q} \geq \tfrac{1}{8}\varepsilon_k^{p+2q} \end{aligned}$$

for r large enough and $k \geq k_o$. This implies that $\sup_{x \in L} |\partial^\beta u_{\varepsilon_k}(x)| \geq \frac{1}{8}\varepsilon_k^{p+2q}$ for $k \geq k_o$, contradicting (2.3) because $\varepsilon_k \to 0$. □

The corresponding result for tempered generalized functions requires some restrictions on the underlying open set Ω. By an n-dimensional box we mean a subset of the form $I_1 \times \ldots \times I_n$ where each I_k is a (finite or infinite) open interval in $\mathbb{R}$.

Proposition 2.5. *If Ω is an n-dimensional box and $U \in \mathcal{G}_\tau(\Omega)$ then*

$$U = 0 \text{ in } \mathcal{G}_\tau(\Omega) \ \Leftrightarrow\ U(\widetilde{x}) = 0 \text{ in } \mathcal{K} \text{ for all } \widetilde{x} \in \widetilde{\Omega}.$$

Proof. $\Rightarrow$: Follows from (the proof of) 2.3.

$\Leftarrow$: That $U \neq 0$ in $\mathcal{G}_\tau(\Omega)$ means

$$\exists \alpha \in \mathbb{N}_o^n\ \forall p > 0\ \exists q > 0\ \forall \eta > 0\ \exists 0 < \varepsilon < \eta: \ \sup_{x \in \Omega} (1 + |x|)^{-p} |\partial^\alpha u_\varepsilon(x)| > \varepsilon^q. \tag{2.4}$$

We claim that

$$\tag{2.5} \begin{gathered} \exists p_o > 0\ \forall p \geq p_o\ \exists q > 0\ \exists s > 0\ \forall \eta > 0\ \exists 0 < \varepsilon < \eta : \\ \sup_{x \in \Omega, |x| \leq \frac{1}{\varepsilon^s}} (1 + |x|)^{-p} |\partial^\alpha u_\varepsilon(x)| > \varepsilon^q. \end{gathered}$$

Indeed, since $(u_\varepsilon)_\varepsilon \in \mathcal{E}_\tau(\Omega)$ there are $p_1 > 0$, $\eta_1 > 0$ such that

$$|\partial^\alpha u_\varepsilon(x)| \leq (1 + |x|)^{p_1} \varepsilon^{-p_1} \quad \forall x \in \Omega,\ \forall 0 < \varepsilon < \eta_1.$$

If we take $p_o > p_1$ then for any $p \geq p_o$ we get

$$\begin{aligned} &\sup_{x \in \Omega, |x| \geq \frac{1}{\varepsilon^s}} (1 + |x|)^{-p} |\partial^\alpha u_\varepsilon(x)| \leq \sup_{x \in \Omega, |x| \geq \frac{1}{\varepsilon^s}} (1 + |x|)^{-p_o} |\partial^\alpha u_\varepsilon(x)| \leq \\ &\leq \sup_{x \in \Omega, |x| \geq \frac{1}{\varepsilon^s}} (1 + |x|)^{-p_o + p_1} \varepsilon^{-p_1} \leq \left(1 + \tfrac{1}{\varepsilon^s}\right)^{p_1 - p_o} \varepsilon^{-p_1} \leq \varepsilon^{s(p_o - p_1) - p_1} \end{aligned}$$

for $0 < \varepsilon < \eta_1$. Taking the q from (2.4) belonging to the given p we have

$$\varepsilon^{s(p_o - p_1) - p_1} \leq \varepsilon^q \quad \text{for } s > \frac{q + p_1}{p_o - p_1}.$$

(We can clearly suppose $\eta_1 < 1$.) With this choice,

$$\tag{2.6} \sup_{x \in \Omega, |x| \geq \frac{1}{\varepsilon^s}} (1 + |x|)^{-p} |\partial^\alpha u_\varepsilon(x)| \leq \varepsilon^q \quad (0 < \varepsilon < \eta_1).$$

For $p \geq p_o$, (2.4) together with (2.6) yields

$$\forall \eta > 0\ \exists 0 < \varepsilon < \eta : \sup_{x \in \Omega, |x| \leq \frac{1}{\varepsilon^s}} (1 + |x|)^{-p} |\partial^\alpha u_\varepsilon(x)| > \varepsilon^q,$$

thus verifying (2.5). This shows that for any $p \geq p_o$ there exist $q > 0$, $s > 0$ and sequences $(x_k)_k$ in Ω and $\varepsilon_k \to 0$ such that $|x_k| \leq \frac{1}{\varepsilon_k^s}$ and

$$\tag{2.7} (1 + |x_k|)^{-p} |\partial^\alpha u_{\varepsilon_k}(x_k)| > \varepsilon_k^q.$$

Set $x_\varepsilon := x_k$ for $\varepsilon_{k+1} < \varepsilon \leq \varepsilon_k$, $k \in \mathbb{N}$. Then $|x_\varepsilon| \leq \frac{1}{\varepsilon^s}$ for $\varepsilon_{k+1} < \varepsilon \leq \varepsilon_k$, so $\widetilde{x} := \mathrm{cl}[(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}$ and $\partial^\alpha U(\widetilde{x}) \neq 0$ in $\mathcal{K}$. Again, we distinguish two cases:

(i) If $\alpha = 0$ then $U(\widetilde{x}) \neq 0$ and we are done.

(ii) Assuming $\alpha \neq 0$ we take an α with minimal $|\alpha|$ satisfying (2.4). Without loss of generality we may suppose $\alpha_1 \neq 0$. Set $\beta := (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n)$, $\gamma := (\alpha_1 + 1, \alpha_2, \ldots, \alpha_n)$ and $x = (x_1, x')$ for $x \in \mathbb{R}^n$. By minimality,

$$\tag{2.8} \exists p_2 > 0\ \forall r > 0\ \exists \eta > 0\ \forall 0 < \varepsilon < \eta : \sup_{x \in \Omega} (1 + |x|)^{-p_2} |\partial^\beta u_\varepsilon(x)| \leq \varepsilon^r$$

By the $\mathcal{E}_\tau$-property of $(u_\varepsilon)_\varepsilon$,

$$\tag{2.9} \exists p_3 > 0\ \exists \eta > 0\ \forall 0 < \varepsilon < \eta : \sup_{x \in \Omega} (1 + |x|)^{-p_3} |\partial^\gamma u_\varepsilon(x)| \leq \varepsilon^{-p_3}$$

Take $p = \max(p_o, p_2, p_3)$ in (2.5) and s, q, $\varepsilon_k \to 0$, $(x_k)_k$ (and from these $\widetilde{x}$) corresponding to this p. Then

$$
\begin{aligned}
|\partial^\alpha u_{\varepsilon_k}(y_1, x_{2,k}, \ldots, x_{n,k})| &= |\partial^\alpha u_{\varepsilon_k}(x_k) + \int_{x_{1,k}}^{y_1} \partial^\gamma u_{\varepsilon_k}(\xi, x'_k) d\xi| \geq \\
&\overset{(2.7),(2.9)}{\geq} (1 + |x_k|)^p \varepsilon_k^q - (1 + |(x_{1,k} + \theta(y_1 - x_{1,k}), x'_k)|)^p \varepsilon_k^{-p} |y_1 - x_{1,k}| \\
&\geq (1 + |x_k|)^p \varepsilon_k^q - (2 + |x_k|)^p \varepsilon_k^{-p} |y_1 - x_{1,k}| \ (\text{if } |y_1 - x_{1,k}| < 1) \\
&\geq ((1 + |x_k|)^p - \tfrac{1}{2}(1 + \tfrac{|x_k|}{2})^p) \varepsilon_k^q \geq \tfrac{1}{2}(1 + |x_k|)^p \varepsilon_k^q
\end{aligned} \tag{2.10}
$$

provided $|y_1 - x_{1,k}| \leq (\frac{1}{2})^{p+1} \varepsilon_k^{p+q}$ and $(y_1, x_{2,k}, \ldots, x_{n,k}) \in \Omega$. Since Ω is a box, there exists some k_o such that for each $k \geq k_o$, either $(x_{1,k} + (\frac{1}{2})^{p+1} \varepsilon_k^{p+q}, x'_k)$ or $(x_{1,k} - (\frac{1}{2})^{p+1} \varepsilon_k^{p+q}, x'_k)$ is in Ω. For each $k \geq k_o$, choose one of these values which is in Ω and denote it by $\overline{x}_k$. Then

$$
\begin{aligned}
|\partial^\beta u_{\varepsilon_k}(\overline{x}_k)| &= |\partial^\beta u_{\varepsilon_k}(x_k) + \int_{x_{1,k}}^{\overline{x}_{1,k}} \partial^\alpha u_{\varepsilon_k}(\xi, x'_k) d\xi| \overset{(2.8),(2.10)}{\geq} -(1 + |x_k|)^p \varepsilon_k^r + \\
&+ |\overline{x}_{1,k} - x_{1,k}| \tfrac{1}{2}(1 + |x_k|)^p \varepsilon_k^q = (1 + |x_k|)^p (-\varepsilon_k^r + (\tfrac{1}{2})^{p+2} \varepsilon_k^{p+2q})
\end{aligned} \tag{2.11}
$$

Now for k large enough we have

$$
\begin{aligned}
(1 + |x_k|)^p &\geq (1 + |\overline{x}_k| - |\overline{x}_k - x_k|)^p = (1 - (\tfrac{1}{2})^{p+1} \varepsilon_k^{p+q} + |\overline{x}_k|)^p \geq \\
&\geq (\tfrac{1}{2} + |\overline{x}_k|)^p \geq (\tfrac{1}{2})^p (1 + |\overline{x}_k|)^p
\end{aligned}
$$

In addition, we can choose r and k so large that $\varepsilon_k^r < (\frac{1}{2})^{p+3} \varepsilon_k^{p+2q}$, so that by (2.11) for sufficiently large k and r we get

$$
|\partial^\beta u_{\varepsilon_k}(\overline{x}_k)| \geq (1 + |\overline{x}_k|)^p (\frac{1}{2})^{2p+3} \varepsilon_k^{p+2q}.
$$

But this contradicts (2.8). □

Next, we are going to extend the range of applicability of 2.5 from boxes to a wider class of open subsets of $\mathbb{R}^n$. We first note the following immediate consequence of the basic definitions:

Lemma 2.6. *Let Ω' be an open subset of $\mathbb{R}^n$ and $f : \Omega' \to \Omega$ a diffeomorphism such that both f and f^{-1} are $\mathcal{O}_M$-functions. Then $U \in \mathcal{G}_\tau(\Omega)$ is 0 in $\mathcal{G}_\tau(\Omega)$ iff $U \circ f = 0$ in $\mathcal{G}_\tau(\Omega')$.* □

In the situation of Lemma 2.6, we shall say that Ω is $\mathcal{O}_M$-diffeomorphic with Ω'. Thus from 2.5 and 2.6 we conclude

Theorem 2.7. *If $\Omega \subseteq \mathbb{R}^n$ is $\mathcal{O}_M$-diffeomorphic with some open box and $U \in \mathcal{G}_\tau(\Omega)$ then*

$$
U = 0 \text{ in } \mathcal{G}_\tau(\Omega) \ \Leftrightarrow \ U(\widetilde{x}) = 0 \text{ in } \mathcal{K} \text{ for all } \widetilde{x} \in \widetilde{\Omega}.
$$

□

That the conclusion of 2.7 cannot be generalized to arbitrary open sets is demonstrated by the following

Example 2.8. For $x \in \mathbb{R}$, by $\lceil x \rceil$ we denote the least integer $\geq x$. Now take

$$\Omega = \bigcup_{n=2}^{\infty} \left(n - \frac{1}{n^n}, n + \frac{1}{n^n} \right) =: \bigcup_{n=2}^{\infty} I_n$$

For $\varepsilon > 0$ set $n_\varepsilon = \lceil \frac{1}{\varepsilon} \rceil$. We define a smooth function u_ε on Ω by

$$u_\varepsilon(x) = \begin{cases} x - n_\varepsilon & x \in I_{n_\varepsilon} \\ 0 & x \in \Omega \setminus I_{n_\varepsilon} \end{cases}$$

$(u_\varepsilon)_\varepsilon \in \mathcal{E}_\tau(\Omega)$ since it is bounded in all derivatives, uniformly in ε. Hence $U := \mathrm{cl}[(u_\varepsilon)_\varepsilon] \in \mathcal{G}_\tau(\Omega)$ and we show that $U \neq 0$: Suppose that $(u_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\Omega)$. Then

$$\text{(2.12)} \qquad \exists p > 0\ \forall q > 0\ \exists \eta > 0\ \forall x \in \Omega: \ |u'_\varepsilon(x)| \leq (1 + |x|)^p \varepsilon^q \quad (0 < \varepsilon < \eta).$$

Setting $x = n_\varepsilon$ in (2.12) we get $1 \leq (1 + n_\varepsilon)^p \varepsilon^q \leq 2^p (\frac{1}{\varepsilon})^p \varepsilon^q$ for arbitrary q and small ε, which is absurd. Nevertheless, all pointvalues of U on generalized points are 0: Indeed, for arbitrary $q > 0$ we even have $\sup_{x \in \Omega} |u_\varepsilon(x)| \leq n_\varepsilon^{-n_\varepsilon} \leq \varepsilon^q$ for $\varepsilon < \min(\frac{1}{q}, 1)$. So, in particular, if $\widetilde{x} = \mathrm{cl}[(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}$ then $|u_\varepsilon(x_\varepsilon)| < \varepsilon^q$ for arbitrary q and sufficiently small ε, i.e. $U(\widetilde{x}) = 0$ in $\mathcal{K}$.

In applications it is often necessary to consider the following 'mixed' variant of Colombeau algebras:

Definition 2.9. Let $\Omega \subseteq \mathbb{R}^m$, $\Omega' \subseteq \mathbb{R}^n$ be open sets. We define

$$\begin{aligned} \widetilde{\mathcal{E}}_\tau(\Omega \times \Omega') = \{ &(u_\varepsilon)_\varepsilon \in (\mathcal{C}^\infty(\Omega \times \Omega'))^I : \forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_o^{m+n}\ \exists p > 0 \\ &\sup_{x \in K} \sup_{y \in \Omega'} (1 + |y|)^{-p} |\partial^\alpha u_\varepsilon(x, y)| = O(\varepsilon^{-p})\ (\varepsilon \to 0) \} \\ \widetilde{\mathcal{N}}_\tau(\Omega \times \Omega') = \{ &(u_\varepsilon)_\varepsilon \in (\mathcal{C}^\infty(\Omega \times \Omega'))^I : \forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_o^{m+n}\ \exists p > 0 \\ &\forall q > 0\ \sup_{x \in K} \sup_{y \in \Omega'} (1 + |y|)^{-p} |\partial^\alpha u_\varepsilon(x, y)| = O(\varepsilon^{q})\ (\varepsilon \to 0) \} \\ \widetilde{\mathcal{G}}_\tau(\Omega \times \Omega') &= \widetilde{\mathcal{E}}_\tau(\Omega \times \Omega') / \widetilde{\mathcal{N}}_\tau(\Omega \times \Omega'). \end{aligned}$$

Thus, the elements of $\widetilde{\mathcal{G}}_\tau(\Omega \times \Omega')$ satisfy $\mathcal{G}$-bounds in the x-variables and $\mathcal{G}_\tau$-bounds in the y-variables. A pointvalue characterization for elements of such algebras can immediately be derived from the results achieved so far:

Theorem 2.10. *Let $\Omega \subseteq \mathbb{R}^m$ be an open set and let $\Omega' \subseteq \mathbb{R}^n$ be an $\mathcal{O}_M$-diffeomorphic image of some open box. Then for any $U \in \widetilde{\mathcal{G}}_\tau(\Omega \times \Omega')$ we have*

$$U = 0 \text{ in } \widetilde{\mathcal{G}}_\tau(\Omega \times \Omega') \ \Leftrightarrow \ U(\widetilde{x}, \widetilde{y}) = 0 \quad \forall \widetilde{x} \in \widetilde{\Omega}_c,\ \forall \widetilde{y} \in \widetilde{\Omega}'.$$

P r o o f. Clearly, $U = 0$ in $\widetilde{\mathcal{G}}_\tau(\Omega \times \Omega')$ implies that all generalized pointvalues are 0 by the same reasoning as in 2.3. Conversely, take some open box B with $\overline{B} \subset\subset \Omega$ and set

$V := U|_{B\times\Omega'}$. If $(\widetilde{x},\widetilde{y}) \in \widetilde{B}\times\widetilde{\Omega'}$, then $\widetilde{x} \in \widetilde{\Omega}_c$, so $V(\widetilde{x},\widetilde{y}) = U(\widetilde{x},\widetilde{y}) = 0$ by assumption. Now from 2.7 we conclude that $V = 0$ in $\mathcal{G}_\tau(B \times \Omega')$. Thus, if K is a compact subset of B we obtain the $\mathcal{N}_\tau$-estimates for U on $K \times \Omega'$ from the $\mathcal{N}_\tau$-estimates for V on $B \times \Omega'$. Since every $K \subset\subset \Omega$ is included in a finite union of relatively compact open boxes in Ω, $U = 0$. $\square$

Note that by a similar argument one can deduce 2.4 from 2.7.

Remark 2.11. An analogous pointvalue characterization is available for elements of the algebra $\mathcal{F}(\Omega)$ introduced by Egorov in [3]. $\mathcal{F}(\Omega)$ is defined as the factor algebra $\mathcal{A}(\Omega)/\mathcal{I}(\Omega)$ where $\mathcal{A}(\Omega)$ is the algebra of sequences $(f_k)_k$ of smooth functions on Ω modulo the ideal consisting of those sequences that eventually vanish on each $K \subset\subset \Omega$. The corresponding generalized numbers $\widetilde{\mathbb{C}}$ are defined as $\bar{\mathbb{C}}^{\mathbb{N}}/\mathcal{I}$ where $\bar{\mathbb{C}}$ is the one-point compactification of $\mathbb{C}$ and $\mathcal{I} = \{(c_k)_k \in \bar{\mathbb{C}}^{\mathbb{N}} : \exists N \in \mathbb{N} : c_k = 0\ \forall k \geq N\}$. Inserting $x \in \Omega$ componentwise into some $f \in \mathcal{F}(\Omega)$ yields a well-defined pointvalue $f(x) \in \widetilde{\mathbb{C}}$. However, example 2.1 with $\varepsilon = \frac{1}{k}$ demonstrates that elements of $\mathcal{F}(\Omega)$ are not uniquely determined by their pointvalues. Thus on

$$\Omega_c := \{(x_k)_k \in \Omega^{\mathbb{N}} : \exists K \subset\subset \Omega\ \exists N \in \mathbb{N} : x_k \in K\ \forall k \geq N\}$$

we introduce an equivalence relation by $(x_k)_k \sim (y_k)_k$ if for some $N \in \mathbb{N}$ $x_k = y_k$ for all $k \geq N$. Then $\widetilde{\Omega}_c := \Omega_c/\sim$ is the set of compactly supported points in the $\mathcal{F}(\Omega)$-setting. Clearly, if $f = \mathrm{cl}[(f_k)_k] \in \mathcal{F}(\Omega)$ and $\widetilde{x} = \mathrm{cl}[(x_k)_k] \in \widetilde{\Omega}_c$ then $f(\widetilde{x}) := \mathrm{cl}[(f_k(x_k))_k]$ is a well-defined element of $\widetilde{\mathbb{C}}$. Then we have

$$f = 0 \text{ in } \mathcal{F}(\Omega) \ \Leftrightarrow\ f(\widetilde{x}) = 0 \text{ in } \widetilde{\mathbb{C}} \text{ for all } \widetilde{x} \in \widetilde{\Omega}_c.$$

Indeed, necessity is immediate from the definitions. Conversely, if $f = \mathrm{cl}[(f_k)_k] \neq 0$ then

$$\exists K \subset\subset \Omega\ \forall k \in \mathbb{N}\ \exists n_k \geq k\ \exists x_{n_k} \in K : f_{n_k}(x_{n_k}) \neq 0.$$

Set $y_m = x_{n_k}$ for $n_k \leq m < n_{k+1}$. Then $\widetilde{y} := \mathrm{cl}[(y_m)_m] \in \widetilde{\Omega}_c$ and $f(\widetilde{y}) \neq 0$, as required.

Finally, we consider some immediate implications of this new pointvalue concept on the theory of Colombeau algebras. In a sense, generalized points are the 'right' adaptation of classical pointvalues to Colombeau functions as they take into account the basic structure of the factor algebras under consideration. Thus, direct generalizations of results from classical analysis become possible:

Example 2.12. *Mean value theorem*: Let I be an open interval in $\mathbb{R}$, $U \in \mathcal{G}(I)$ and let $\widetilde{x},\widetilde{y} \in \widetilde{I}_c$, $\widetilde{x} \leq \widetilde{y}$ (meaning that there exist representatives $(x_\varepsilon)_\varepsilon$, $(y_\varepsilon)_\varepsilon$ with $x_\varepsilon \leq y_\varepsilon$ for all ε). Then there exists some $\widetilde{\xi} \in \widetilde{I}_c$ with $\widetilde{x} \leq \widetilde{\xi} \leq \widetilde{y}$ such that $U(\widetilde{y}) - U(\widetilde{x}) = U'(\widetilde{\xi})(\widetilde{y} - \widetilde{x})$. (In particular, one may consider $x, y \in I$). This follows immediately from a componentwise application of the classical mean value theorem.

An important consequence of theorems 2.4 and 2.7 is that they allow a geometric interpretation of generalized functions by identifying any element U of $\mathcal{G}(\Omega)$ with its graph

$$\Gamma_U := \{(\widetilde{x}, U(\widetilde{x})) : \widetilde{x} \in \widetilde{\Omega}_c\}$$

(and analogously for $U \in \mathcal{G}_\tau(\Omega)$). Such a description enables a generalization of classical geometrical methods (e.g. action of transformation groups, cf. [6]) to elements of Colombeau algebras. This in turn can be utilized for extending the range of applicability of group analysis of differential equations to include nonlinear operations on distributional solutions ([7]). Moreover, in many cases a transfer of methods from classical analysis is greatly facilitated by (or even enabled through) making use of pointvalue arguments (e.g. flow-properties of generalized ODEs, cf. [4]).

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