PS Gewöhnliche Differentialgleichungen 1

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1. Consider the case of a stone dropped from the height h. Denote by r the distance of the stone from the surface. The initial condition reads $r(0) = h$, $\dot{r}(0) = 0$. The equation of motion reads

$$\ddot{r} = -\frac{\gamma M}{(R+r)^2} \qquad \text{(exact model)}$$

respectively

$$\ddot{r} = -g \qquad \text{(approximate model)},$$

where $g = \gamma M/R^2$ and R, M are the radius, mass of the earth, respectively.

   (a) Transform both equations into a first-order system.

   (b) Compute the solution to the approximate system corresponding to the given initial condition. Compute the time it takes for the stone to hit the surface ($r = 0$).

   (c) Assume that the exact equation also has a unique solution corresponding to the given initial condition. What can you say about the time it takes for the stone to hit the surface in comparison to the approximate model? Will it be longer or shorter? Estimate the difference between the solutions in the exact and in the approximate case. (Hints: You should not compute the solution to the exact equation! Look at the minimum, maximum of the force.)

   (d) Grab your physics book from high school and give numerical values for the case $h = 10m$.

2. Consider again the exact model from the previous problem and write

$$\ddot{r} = -\frac{\gamma M \varepsilon^2}{(1+\varepsilon r)^2}, \qquad \varepsilon = \frac{1}{R}.$$

It can be shown that the solution $r(t) = r(t,\varepsilon)$ to the above initial conditions is C^∞ (with respect to both t and ε). Show that

$$r(t) = h - g(1 - 2\frac{h}{R})\frac{t^2}{2} + O(\frac{1}{R^4}), \qquad g = \frac{\gamma M}{R^2}.$$

(Hint: Insert $r(t,\varepsilon) = r_0(t) + r_1(t)\varepsilon + r_2(t)\varepsilon^2 + r_3(t)\varepsilon^3 + O(\varepsilon^4)$ into the differential equation and collect powers of ε. Then solve the corresponding differential equations for $r_0(t), r_1(t), \ldots$ and note that the initial conditions follow from $r(0,\varepsilon) = h$ respectively $\dot{r}(0,\varepsilon) = 0$.)

3. Classify the following differential equations. Is the eqation linear, autonomous? What is its order?

(a) $y'(x) + y(x) = 0$.
(b) $\frac{d^2}{dt^2}u(t) = \sin(u(t))$.
(c) $y(t)^2 + 2y(t) = 0$.
(d) $\frac{\partial^2}{\partial x^2}u(x,y) + \frac{\partial^2}{\partial y^2}u(x,y) = 0$.
(e) $\dot{x} = -y$, $\dot{y} = x$.

4. Which of the following differential equations are linear?

   (a) $y'(x) = \sin(x)y + \cos(y)$.
   (b) $y'(x) = \sin(y)x + \cos(x)$.
   (c) $y'(x) = \sin(x)y + \cos(x)$.

5. Transform the following differential equations into first-order systems.

   (a) $\ddot{x} + t\,\sin(\dot{x}) = x$.
   (b) $\ddot{x} = -y$, $\ddot{y} = x$.

   The last system is linear. Is the corresponding first-order system also linear? Is this always the case?

6. Solve the following differential equations:

   (a) $\dot{x} = \sin(t)x$.
   (b) $\dot{x} = g(t)\tan(x)$.

7. A certain species of bacteria grows according to

$$\dot{N}(t) = \kappa N(t), \qquad N(0) = N_0,$$

   where $N(t)$ is the amount of bacteria at time t, $\kappa > 0$ is the growth rate, and N_0 is the initial amount. If there is only space for $N_{\max}$ bacteria, this has to be modified according to

$$\dot{N}(t) = \kappa(1 - \frac{N(t)}{N_{\max}})N(t), \qquad N(0) = N_0.$$

   Solve both equations, assuming $0 < N_0 < N_{\max}$ and discuss the solutions. What is the behavior of $N(t)$ as $t \to \infty$?

8. Consider the free fall with air resistance modeled by

$$\ddot{x} = -\eta\dot{x} - g, \qquad \eta > 0.$$

   Solve this equation (Hint: Introduce the velocity $v = \dot{x}$ as new independent variable). Is there a limit to the speed the object can attain? If yes, find it. Consider the case of a parachutist. Suppose the chute is opened at a certain time $t_0 > 0$. Model this situation by assuming $\eta = \eta_1$ for $0 < t < t_0$ and $\eta = \eta_2 > \eta_1$ for $t > t_0$ and match the solutions at t_0. What does the solution look like?

9. Transform the differential equation

$$t^2\ddot{x} + 3t\dot{x} + x = \frac{2}{t}$$

to the new coordinates $y = x$, $s = \ln(t)$. (Hint: You are *not* asked to solve it.)

10. Suppose you have an incoming electromagnetic wave along the y-axis which should be focused on a receiver sitting at the origin $(0, 0)$. What is the optimal shape for the mirror?

(Hint: An incoming ray, hitting the mirror at (x, y) is given by

$$R_{\text{in}}(t) = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} t, \quad t \in (-\infty, 0].$$

At (x, y) it is reflected and moves along

$$R_{\text{rfl}}(t) = \begin{pmatrix} x \\ y \end{pmatrix} (1 - t), \quad t \in [0, 1].$$

The laws of physics require that the angle between the tangent of the mirror and the incoming respectively reflected ray must be equal. Considering the scalar products of the vectors with the tangent vector this yields

$$\frac{1}{\sqrt{1+u^2}} \begin{pmatrix} 1 \\ u \end{pmatrix} \begin{pmatrix} 1 \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ y' \end{pmatrix}, \quad u = \frac{y}{x},$$

which is the differential equation for $y = y(x)$ you have to solve.)

11. (*) Show that the nonlinear boundary value problem

$$y''(x) + y(x)^2 = 0, \qquad y(0) = y(1) = 0,$$

has a unique nontrivial solution. Assume that the initial value problem $y(x_0) = x_0$, $y'(x_0) = y_1$ has a unique solution.

- Show that a nontrivial solution of the boundary value problem must satisfy $y'(0) = p_0 > 0$.
- If a solution satisfies $y'(x_0) = 0$, then the solution is symmetric with respect to this point: $y(x) = y(x_0 - x)$. (Hint: Uniqueness.)
- Solve the initial value problem $y(0) = 0$, $y'(0) = p_0 > 0$ as follows: Set $y' = p(y)$ and derive a first-order equation for $p(y)$. Solve this equation for $p(y)$ and then solve the equation $y' = p(y)$. (Note that this works for any equation of the type $y'' = f(y)$.)
- Does the solution found in the previous item attain $y'(x_0) = 0$ at some x_0? What value should x_0 have for $y(x)$ to solve our boundary value problem?
- Can you find a value for p_0 in terms of special functions?

12. Let x be a solution of $\dot{x} = f(x)$, $x(0) = x_0$ which satisfies $\lim_{t\to\infty} x(t) = x_1$. Show that $\lim_{t\to\infty} \dot{x}(t) = 0$ and $f(x_1) = 0$.

13. Discuss the equation $\dot{x} = x^2 - \frac{t^2}{1+t^2}$.
    - Make a numerical analysis.
    - Show that there is a unique solution which asymptotically approaches the line $x = 1$.
    - Show that all solutions below this solution approach the line $x = -1$.
    - Show that all solutions above go to ∞ in finite time.

14. Show that $f \in C^1(\mathbb{R})$ is locally Lipschitz continuous. In fact, show that

$$|f(y) - f(x)| \le \sup_{\varepsilon\in[0,1]} |f'(x + \varepsilon(y - x))||x - y|.$$

Generalize this result to $f \in C^1(\mathbb{R}^m, \mathbb{R}^n)$.

15. Are the following functions Lipschitz continuous near 0? If yes, find a Lipschitz constant for some interval containing 0.
    (a) $f(x) = \frac{1}{1-x^2}$.
    (b) $f(x) = |x|^{1/2}$.
    (c) $f(x) = x^2 \sin(\frac{1}{x})$.

16. Consider the initial value problem $\dot{x} = x^2$, $x(0) = x_0 > 0$. What is the maximal value for T_0 (as a function of x_0) according to Theorem 2.2 respectivley Theorem 2.5? What maximal value do you get from the explicit solution? (Hint: Compute T_0 as a function of δ and find the optimal δ.)

17. Suppose $\psi(t)$ satisfies

$$\psi(t) \le \alpha(t) + \int_0^t \beta(s)\psi(s)ds$$

with $\beta(t) \ge 0$. Show that

$$\psi(t) \le \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)dr\right) ds. \tag{1}$$

Moreover, if in addition $\alpha(s) \le \alpha(t)$ for $s \le t$, then

$$\psi(t) \le \alpha(t) \exp\left(\int_0^t \beta(s)ds\right).$$

(Hint: Denote the right hand side of (1) by $\phi(t)$ and show that it satisfies $\phi(t) = \alpha(t) + \int_0^t \beta(s)\phi(s)ds$. Then consider $\Delta(t) = \psi(t) - \phi(t)$ which is continuous and satisfies $\Delta(t) \le \int_0^t \beta(s)\Delta(s)ds$.)

18. Find functions $f(t, x) = f(x)$ and $g(t, x) = g(x)$ such that the inequality

$$|x(t) - y(t)| \le |x_0 - y_0| \, e^{L|t-t_0|} + \frac{M}{L}(e^{L|t-t_0|} - 1),$$

from Theorem 2.8 becomes an equality.

19. Show that in the one dimensional case, we have
$$\frac{\partial \phi}{\partial x}(t,x) = \exp\left(\int_{t_0}^{t} \frac{\partial f}{\partial x}(s,\phi(s,x))ds\right).$$

20. Consider a first-order autonomous system in $\mathbb{R}^n$ with $f(x)$ Lipschitz. Show that $x(t)$ is a solution if and only if $x(t-t_0)$ is. Use this and uniqueness to show that for two maximal solutions $x_j(t)$, $j=1,2$, the images $\gamma_j = \{x_j(t) | t \in I_j\} \subset \mathbb{R}^n$ either coincide or are disjoint.

21. Consider a first-order autonomous equation in $\mathbb{R}^1$ with $f(x)$ Lipschitz. Suppose $f(0)=f(1)=0$. Show that solutions starting in $[0,1]$ cannot leave this interval. What is the maximal interval of definition (T_-,T_+) for solutions starting in $[0,1]$? Does such a solution have a limit as $t \to T_\pm$?

22. Consider a first-order equation in $\mathbb{R}^1$ with $f(t,x)$ defined on $\mathbb{R}\times\mathbb{R}$. Suppose $x\,f(t,x) < 0$ for $|x| > R$. Show that all solutions exist for all $t > 0$.

23. Show that the space of n by n matrices $\mathbb{C}^{n\times n}$ together with the matrix norm is a Banach space. In particular, show that a sequence of matrices converges if and only if all matrix entries converge. (Hint: Show that the matrix entries a_{jk} of A satisfy $\max_{j,k} |a_{jk}| \le \|A\|$ and $\|A\| \le n \max_{j,k} |a_{jk}|$.)

24. Show that the matrix norm satisfies
$$\|AB\| \le \|A\| \|B\|.$$
(This shows that $\mathbb{C}^{n\times n}$ is even a **Banach algebra**.) Conclude $\|A^j\| \le \|A\|^j$.

25. Show that the matrix product is continuous with respect to the matrix norm. That is, if $A_j \to A$ and $B_j \to B$ we have $A_jB_j \to AB$. (Hint: Previous problem).

26. (a) Compute $\exp(A)$ for
$$A = \begin{pmatrix} a+d & b \\ c & a-d \end{pmatrix}.$$
(b) Is there a real matrix A such that
$$\exp(A) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix}, \quad \alpha, \beta > 0?$$

27. Solve the systems corresponding to the following matrices:
$$(a)\; A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \qquad (b)\; A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

28. Solve
$$\dot{x} = -y - t, \quad \dot{y} = x + t, \quad x(0) = 1, y(0) = 0.$$

29. Find a two by two matrix such that $x(t) = (\sinh(t), \mathrm{e}^t)$ is a solution.

30. Which of the following functions

    (a) $x(t) = (3\mathrm{e}^t + \mathrm{e}^{-t}, \mathrm{e}^{2t})$.
    (b) $x(t) = (3\mathrm{e}^t + \mathrm{e}^{-t}, \mathrm{e}^{t})$.
    (c) $x(t) = (3\mathrm{e}^t + \mathrm{e}^{-t}, t\mathrm{e}^{t})$.
    (d) $x(t) = (3\mathrm{e}^t, t^2\mathrm{e}^{t})$.

    can be solutions of a first-order autonomous homogeneous system?

31. Let A be a real 2 by 2 matrix. Then the eigenvalues can be expressed in terms of the determinant $D = \det(A)$ and the trace $T = \mathrm{tr}(A)$. In particular, (T, D) can take all possible values in $\mathbb{R}^2$ if A ranges over all possible matrices in $\mathbb{R}^{2\times 2}$. Split the (T, D) plane into regions in which the various cases discussed in the lecture occure.

32. Solve the equation
$$\ddot{x} + \omega_0^2 x = \cos(\omega t), \quad \omega_0, \omega > 0.$$
    Discuss the behavior of solutions as $t \to \infty$.

33. Verify that the solution of the inhomogeneous equation
$$x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1\dot{x} + c_0 x = g(t)$$
    is given by
$$x(t) = x_h(t) + \int_0^t u(t-s)g(s)ds,$$
    where $x_h(t)$ is an arbitrary solution of the homogenous equation and $u(t)$ is the solution of the homogeneous equation corresponding to the initial condition $u(0) = \dot{u}(0) = \cdots = u^{(n-2)}(0) = 0$ and $u^{(n-1)}(0) = 1$.

34. Derive Taylor's formula with remainder
$$x(t) = \sum_{j=0}^{n} \frac{x^{(j)}(t_0)}{j!}(t-t_0)^j + \frac{1}{n!}\int_{t_0}^t x^{(n+1)}(s)(t-s)^n ds$$
    for $x \in C^{n+1}$ from the previous problem.

35. Look at the second-order equation
$$\ddot{x} + c_1\dot{x} + c_0 x = g$$
    and let α_1, α_2 be the corresponding eigenvalues (not necessarily distinct). Show that the equation can be factorized as
$$\ddot{x} + c_1\dot{x} + c_0 x = \left(\frac{d}{dt} - \alpha_2\right)\left(\frac{d}{dt} - \alpha_1\right) x.$$
    Hence the equation can be reduced to solving two first order equations
$$\left(\frac{d}{dt} - \alpha_2\right) y = g, \qquad \left(\frac{d}{dt} - \alpha_1\right) x = y.$$
    Use this to prove Theorem 3.4 in the case $n = 2$. What can you say about the structure of the solution if $g(t) = p(t)\mathrm{e}^{\beta t}$, where $p(t)$ is a polynomial.

36. Compute $\Pi(t, t_0)$ for the system
$$A(t) = \begin{pmatrix} t & 0 \\ 1 & t \end{pmatrix}.$$

37. Show that if $\limsup_{t\to\infty} \int_{t_0}^t \mathrm{tr}(A(s))ds = \infty$, then $\dot{x} = A(t)x$ has an unbounded solution.

38. Show
$$\|\Pi(t, t_0)\| \le \mathrm{e}^{|\int_{t_0}^t \|A(s)\| ds|}. \qquad (2)$$
(Hint: Generalized Gronwall.)

39. Suppose
$$\int_0^\infty \|A(t)\| dt < \infty.$$
Show that every solution $x(t)$ of $\dot{x} = A(t)x$ converges to some limit:
$$\lim_{t\to\infty} x(t) = x_\infty.$$
(Hint: First show that all solutions are bounded and then use the corresponding integral equation.)

40. Consider the inhomogeneous equation
$$\dot{x}(t) = A(t)x(t) + g(t),$$
where both $A(t)$ and $g(t)$ are periodic of period T. Show that this equation has a unique periodic solution of period T if 1 is not an eigenvalue of the monodromy matrix $M(t_0)$. (Hint: Note that $x(t)$ is periodic if and only if $x(T) = x(0)$ and use the variation of constants formula.)

41. Make a power series ansatz for the following equations:
   (a) $w' + w = z, \qquad w(0) = w_0.$
   (b) $w' + w^2 = z^2, \qquad w(0) = w_0.$
   (c) $w' + w = \frac{1}{1-z}, \qquad w(0) = w_0.$

42. Try to find a solution of the initial value problem
$$w'' = (z^2 - 1)w, \qquad w(0) = 1,\ w'(0) = 0,$$
by using the power series method from above. Can you find a closed form for the solution?

43. Consider the first-order liner inhomogeneous difference equation
$$x(n+1) - f(n)x(n) = g(n), \quad f(n) \neq 0.$$
Show that the solution of the homogeneous equation ($g = 0$) is given by
$$x_h(n) = x(0) \begin{cases} \prod_{j=0}^{n-1} f(j) & \text{for } n > 0 \\ 1 & \text{for } n = 0 \\ \prod_{j=n}^{-1} f(j)^{-1} & \text{for } n < 0 \end{cases}.$$

Use a variation of constants ansatz for the inhomogeneous equation and show that the solution is given by

$$x(n) = x_h(n) + \begin{cases} x_h(n) \sum_{j=0}^{n-1} \frac{g(j)}{x_h(j+1)} & \text{for } n > 0 \\ 0 & \text{for } n = 0 \\ -x_h(n) \sum_{j=n}^{-1} \frac{g(j)}{x_h(j+1)} & \text{for } n < 0 \end{cases}.$$

44. The Legendre equation is given by

$$(1 - z^2)w'' - 2zw' + n(n + 1)w = 0.$$

Make a power series ansatz at $z = 0$ and show that there is a polynomial solution $p_n(z)$ if $n \in \mathbb{N}_0$. What is the order of $p_n(z)$?

45. Show that

$$q_2(x)y'' + q_1(x)y' + q_0(x)y$$

can be written as

$$\frac{1}{r(x)} \left((p(x)y')' + q(x)y\right).$$

Find r, p, q in terms of q_0, q_1, q_2.

Write the Bessel and Legendre equations in this form.

46. Find conditions for the initial values $u(x)$ and $v(x)$ such that

$$u(t, x) = \sum_{n=1}^{\infty} \left(c_{1,n} \cos(cn\pi t) + \frac{c_{2,n}}{cn\pi} \sin(cn\pi t)\right) \sin(n\pi x),$$

is indeed a solution (i.e., such that interchanging the order of summation and differentiation is admissible). (Hint: The decay of the Fourier coefficients is related to the smoothness of the function.)

47. Prove the parallelogram law

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2.$$

48. Show that every compact linear operator is bounded and that the product of a bounded and a compact operator is compact (compact operators form an ideal).

49. Show that if A is bounded, then every eigenvalue α satisfies $|\alpha| \leq \|A\|$.