# PS Harmonische Analysis

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### WS2024/25

Note: References refer to the lecture notes.

1. Show that for $n, m \in \mathbb{N}$ we have

$$2 \int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

   Conclude that the Fourier sine coefficients $g_n$ of

$$g(x) = \sum_{n=1}^\infty g_n \sin(n\pi x).$$

   are given by

$$g_n = 2 \int_0^1 \sin(n\pi x) g(x) dx.$$

   provided the sum converges uniformly.

2. Use the method of separation of variables as described in the notes to solve the **heat equation**

$$\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x)$$

   with boundary conditions $u(t, 0) = u_0$, $u(t, 1) = u_1$ and initial condition $u(0, x) = u(x)$. It models the temperature distribution of a thin wire whose edges are kept at a fixed temperature $u_0$ and $u_1$. What can you say about $\lim_{t \to \infty} u(t, x)$? (Hint: If $u(x, t)$ solves the heat equation, so does $u(x, t) + a + bx$. Use this to reduce the boundary conditions to the case $u_0 = u_1 = 0$.)

3. Show that $\ell^2(\mathbb{N})$ is a Hilbert space.

4. Prove the **parallelogram law**

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

   for $f, g$ in some Hilbert space $\mathfrak{H}$.

5. Let

$$\ell^p(\mathbb{N}) := \{a := (a_j)_{j \in \mathbb{N}} \mid \|a\|_p < \infty\},$$

   where

$$\|a\|_p = \begin{cases} \left( \sum_{j \in \mathbb{N}} |a_j|^p \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{j \in \mathbb{N}} |a_j|, & p = \infty. \end{cases}$$

   Show that $\ell^p(\mathbb{N})$, $1 \leq p \leq \infty$, is a Hilbert space if and only if $p = 2$. (Hint: Previous problem.)

6. Suppose $\mathfrak{Q}$ is a complex vector space. Let $s(f, g)$ be a sesquilinear form on $\mathfrak{Q}$ and $q(f) := s(f, f)$ the associated quadratic form. Prove the **parallelogram law**

$$q(f + g) + q(f - g) = 2q(f) + 2q(g)$$

and the **polarization identity**

$$s(f, g) = \frac{1}{4} \left( q(f + g) - q(f - g) + \mathrm{i}\, q(f - \mathrm{i}g) - \mathrm{i}\, q(f + \mathrm{i}g) \right).$$

Show that $s(f, g)$ is symmetric if and only if $q(f)$ is real-valued.

Note, that if $\mathfrak{Q}$ is a real vector space, then the parallelogram law is unchanged but the polarization identity in the form $s(f, g) = \frac{1}{4}(q(f + g) - q(f - g))$ will only hold if $s(f, g)$ is symmetric.

7. Let $\{u_j\}_{j=0}^{\infty} \subset \mathfrak{H}$ be a countable orthonormal set and $f \in \mathfrak{H}$. Show that

$$f_n = \sum_{j=0}^{n} \langle u_j, f \rangle u_j$$

is a Cauchy sequence.

8. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that $f * g \in L^\infty(\mathbb{R}^n)$ with

$$\|f * g\|_\infty \le \|f\|_p \|g\|_q.$$

Show that for $1 < p < \infty$ or $p = 1$ and $g \in C_0(\mathbb{R}^n)$ we even have $f * g \in C_0(\mathbb{R}^n)$.

9. Let $\phi_\varepsilon$ be a symmetric approximate identity on $\mathbb{R}$, that is, $\phi_\varepsilon(x) = \phi_\varepsilon(-x)$. Show that for every bounded measurable function $f$ we have

$$\lim_{\varepsilon \downarrow 0} (\phi_\varepsilon * f)(x) = \frac{f(x+) + f(x-)}{2},$$

at every point $x$ where both right/left sided limits $f(x\pm) := \lim_{\varepsilon \downarrow 0} f(x \pm \varepsilon)$ exist.

Show that the same conclusion holds for integrable functions $f$ if $\phi_\varepsilon$ has compact support, $\mathrm{supp}(\phi_\varepsilon) \subseteq [-s, s]$, and for every $r > 0$ we have $\lim_{\varepsilon \downarrow 0} \sup_{r \le |x| \le s} |\phi_\varepsilon(x)| = 0$.

10. Show that the Landau kernel

$$L_n(x) := \begin{cases} \frac{1}{I_n}(1 - x^2)^n, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases} \qquad I_n = \int_{-1}^{1} (1 - x^2)^n dx$$

is an approximate identity on $\mathbb{R}$ (for $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}$).

Use this to prove the Weierstraß approximation theorem, that every continuous function on $[-\frac{1}{2}, \frac{1}{2}]$ can be uniformly approximated by polynomials.

11. Show that the Poisson kernel

$$P_\varepsilon(x) := \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}$$

is an approximate identity on $\mathbb{R}$.

Show that the Cauchy transform (also Borel transform)

$$F(z) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(\lambda)}{\lambda - z} d\lambda$$

of a real-valued function $f \in L^p(\mathbb{R})$, $1 \le p < \infty$ is analytic in the upper half-plane with imaginary part given by

$$\operatorname{Im}(F(x + \mathrm{i}y)) = (P_y * f)(x).$$

In particular, by Young's inequality $\|\operatorname{Im}(F(.+\mathrm{i}y))\|_p \le \|f\|_p$ and thus also $\sup_{y>0} \|\operatorname{Im}(F(.+\mathrm{i}y))\|_p = \|f\|_p$. Such harmonic functions are said to be in the Hardy space $h^p(\mathbb{C}_+)$.

(Hint: To see analyticity of $F$ use Problem A.1 from the notes plus the estimate

$$\left| \frac{1}{\lambda - z} \right| \le \frac{1}{1 + |\lambda|} \frac{1 + |z|}{|\operatorname{Im}(z)|}.)$$

12. Let $f \in L^1(-\pi, \pi)$ be periodic and $a \in \mathbb{R}$, $n \in \mathbb{Z}$. Show

| $g(x)$ | $\widehat{g}_k$ |
|---|---|
| $f(-x)$ | $\widehat{f}_{-k}$ |
| $f(x)^*$ | $\widehat{f}_{-k}^*$ |
| $f(x + a)$ | $\mathrm{e}^{\mathrm{i}ak} \widehat{f}_k$ |
| $\mathrm{e}^{\mathrm{i}nx} f(x)$ | $\widehat{f}_{k-n}$ |

13. Show the following estimates for the Dirichlet and Fejér kernels: $|D_n(x)| \le \min(2n+1, \frac{\pi}{|x|})$ and $F_n(x) \le \min(n, \frac{\pi^2}{nx^2})$ for $|x| \le \pi$.

14. Show that the Dirichlet kernel satisfies

$$\frac{8}{\pi} \sum_{k=1}^{n} \frac{1}{k} \le \|D_n\|_1 \le 2\pi(1 + \log(2n+1))$$

and note that the harmonic series diverges.

15. Compute the Fourier series of the Dirichlet kernel $D_n$ and the Fejér kernel $F_n$.

16. Compute the Fourier series of $f(x) := |x|$ on $[-\pi, \pi]$. For which $x \in [-\pi, \pi]$ does $S_n(f)(x)$ converge to $f(x)$?

17. Compute the Fourier series of $f(x) := x^2$ on $[-\pi, \pi]$ and use this to show (Basel problem)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(Hint: Evaluate the series at $x = \pi$.)

18. Show that if a sequence of complex numbers $S_n$ converges to $S$, then the sequence of mean values

$$\bar{S}_n := \frac{1}{n} \sum_{k=0}^{n-1} S_k$$

also converges to $S$.

19. Suppose $f$ and $g$ are periodic (with period $2\pi$) integrable functions. Show

$$\widehat{(f * g)}_k = 2\pi \hat{f}_k \hat{g}_k,$$

where

$$(f * g)(x) = \int_{-\pi}^{\pi} f(x - y)g(y)dy.$$

(Hint: Fubini.)

20. Compute the Fourier series of $f(x) := x$ on $[-\pi, \pi]$ and use this to solve again the Basel problem. (Hint: Parseval's relation.)

21. Show that the Hilbert transform $H$ satisfies

$$\widehat{(Hf_k)} = \frac{\text{sign}(k)}{\text{i}} \hat{f}_k$$

for $f$ a trigonometric polynomial. (Hint: To evaluate the integral consider the closed form of the Dirichlet kernel and split off the $+1/2$.)

22. Compute the Fourier series of $f(x) := \text{sign}(x)$. For which $x \in [-\pi, \pi]$ does $S_n(f)(x)$ converge to $f(x)$?

23. Compute the Fourier series of $f(x) := \frac{\pi}{\sin(\alpha\pi)} e^{i\alpha x}$ on $[-\pi, \pi]$ for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. For which $x \in [-\pi, \pi]$ does $S_n(f)(x)$ converge to $f(x)$?

Establish the partial fraction decomposition

$$\pi z \cot(\pi z) = 1 + 2z^2 \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2}, \qquad z \in \mathbb{C} \setminus \mathbb{Z}.$$

24. Show that for $\delta \in \mathbb{R}$ and $f \in L^1(0, \pi)$ we have

$$\lim_{n \to \infty} \int_0^{\pi} f(x) \sin((n + \delta)x)dx = 0.$$

Use this to compute the Dirichlet integral

$$\lim_{R \to \infty} \int_0^R \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

(Hint: To evaluate the Dirichlet integral start from $\int_0^{\pi} D_n(x)dx = \pi$ and observe that $\frac{2}{x} - \frac{1}{\sin(x/2)}$ is continuous on $[-\pi, \pi]$.)

25. Show the summation by parts formula

$$\sum_{j=m}^{n} g_j(\partial f)_j = g_n f_{n+1} - g_{m-1} f_m + \sum_{j=m}^{n} (\partial^* g)_j f_j$$

where,

$$(\partial f)_n = f_{n+1} - f_n, \qquad (\partial^* f)_n = f_{n-1} - f_n,$$

are the forward/backward difference operators.

Suppose the real-valued sequence $a_n$ converges to zero and satisfies $b_n := -(\partial \partial^* a)_n = a_{n+1} + a_{n-1} - 2a_n \geq 0$ for $n \in \mathbb{N}$. Show that

- $\alpha_n := a_{n-1} - a_n \geq 0$ is decreasing and converges to zero,
- $n\alpha_n \to 0$,
- $a_m = \sum_{j=m+1}^{\infty} (j - m) b_j$.

(Hint: To show the second claim use $a_m - a_n \geq (n - m)\alpha_n$ for $n \geq m$. For the third claim use summation by parts.)

26. Let $(a_k)_{k \in \mathbb{N}}$ be a nonnegative monotone decreasing sequence which converges to zero and $(b_k(x))_{k \in \mathbb{N}}$ a sequence of complex-valued functions for which $B_n(x) = \sum_{k=1}^{n-1} b_k(x)$ is uniformly bounded, $|B_n(x)| \leq C$. Show that (Dirichlet criterion) the series

$$\sum_{k=0}^{\infty} a_k b_k(x)$$

converges uniformly.

Conclude that the Fourier cosine/sine series

$$\sum_{k=0}^{\infty} a_k \cos(kx), \qquad \sum_{k=0}^{\infty} a_k \sin(kx)$$

converge uniformly on every compact subinterval $0 < |x| < 2\pi$. (Hint: To show the Dirichlet criterion use summation by parts from Problem 25.)

27. Show that the operator

$$I : L^1(-\pi, \pi) \to C_{per}[-\pi, \pi], \quad f(x) \mapsto F(x) - \widehat{f}_0 x - \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) dx,$$

where $F(x) := \int_0^x f(y) dy$, satisfies

$$\widehat{I(f)}_k = \begin{cases} 0, & k = 0, \\ \dfrac{\widehat{f}_k}{ik}, & k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Show also $\|I(f)\|_\infty \leq \frac{5}{2} \|f\|_1$.

28. Show that $f \in C_{per}^\infty[-\pi, \pi]$ if and only if $|k^m \widehat{f}_k|$ is bounded for all $m \in \mathbb{N}$.

29. Define the Sobolev space

$$H_{per}^s(-\pi, \pi) := \{f \in L^2(-\pi, \pi) \mid |k|^s \widehat{f}_k \in \ell^2(\mathbb{Z})\}.$$

Show that $H_{per}^1(-\pi, \pi) = \{f \in AC_{per}[-\pi, \pi] \mid f' \in L^2(-\pi, \pi)\}$. Show that the Fourier coefficients of $f \in H_{per}^s(-\pi, \pi)$ are summable for $s > \frac{1}{2}$. Show that this fails for $s = \frac{1}{2}$. (Hint: Problem 27.)

30. Show that if $f \in C_{per}^{0,\gamma}[-\pi, \pi]$ is Hölder continuous, then

$$|\widehat{f}_k| \le \frac{[f]_\gamma}{2} \left(\frac{\pi}{|k|}\right)^\gamma, \qquad k \ne 0.$$

Conclude that if $f \in C_{per}^{l,\gamma}[-\pi, \pi]$ (the set of periodic functions which are $C^l$ and for which the highest derivative is Hölder continuous of exponent $\gamma$) we have

$$|\widehat{f}_k| \le \frac{[f^{(l)}]_\gamma}{2} \left(\frac{\pi}{|k|}\right)^{l+\gamma}, \qquad k \ne 0.$$

(Hint: What changes if you replace $\mathrm{e}^{-iky}$ by $\mathrm{e}^{-ik(y+\pi/k)}$ in a Fourier series? Now make a change of variables $y \to y - \pi/k$ in the integral.)

31. Consider the function

$$f(x) := \sum_{j=0}^\infty 2^{-\gamma j} \mathrm{e}^{i 2^j x}, \qquad \gamma > 0.$$

Show:

   (i) $f$ is in the Wiener algebra (and in particular continuous).

   (ii) $f$ is not of bounded variation for $0 < \gamma < 1$.

   (iii) $f$ is not absolutely continuous for $0 < \gamma \le 1$.

   (iv) $f$ is Hölder continuous of exponent $\gamma$ for $0 < \gamma < 1$ but not Lipschitz continuous for $\gamma = 1$.

(Hint for (iv): To estimate $|f(x+\delta) - f(x)|$ split the sum at an index $n$ and use $|\mathrm{e}^{ix} - 1| \le |x|$, $x \in \mathbb{R}$, for the first part and $|\mathrm{e}^{ix} - 1| \le 2$, $x \in \mathbb{R}$, for the second. Now find a suitable value for $n$.)

32. Let $S_n = \sum_{k=0}^n a_k$ and $\bar{S}_n := \frac{1}{n} \sum_{k=0}^{n-1} S_k$. Show that for $m < n$

$$S_n - \bar{S}_{n+1} = \frac{m+1}{n-m} (\bar{S}_{n+1} - \bar{S}_{m+1}) + \frac{1}{n-m} \sum_{k=m+1}^n (S_n - S_k).$$

33. Show that $F^1(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ with

$$\|f\|_p \le (2\pi)^{-\frac{n}{2}(1-\frac{1}{p})} \|f\|_1^{\frac{1}{p}} \|\widehat{f}\|_1^{1-\frac{1}{p}}.$$

Moreover, show that $\mathcal{S}(\mathbb{R}^n) \subset F^1(\mathbb{R}^n)$ and conclude that $F^1(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $p \in [1, \infty)$. (Hint: Use $x^p \le x$ for $0 \le x \le 1$ to show $\|f\|_p \le \|f\|_\infty^{1-1/p} \|f\|_1^{1/p}$.)

34. Suppose $f_j \in L^1(\mathbb{R})$, $j = 1, \ldots, n$ and set $f(x) := \prod_{j=1}^n f_j(x_j)$. Show that $f \in L^1(\mathbb{R}^n)$ with $\|f\|_1 = \prod_{j=1}^n \|f_j\|_1$ and $\widehat{f}(p) = \prod_{j=1}^n \widehat{f}_j(p_j)$.

35. Compute the Fourier transform of the following functions $f : \mathbb{R} \to \mathbb{C}$:

    (i) $f(x) = \chi_{(-1,1)}(x)$.      (ii) $f(x) = \frac{e^{-k|x|}}{k}$,    $\mathrm{Re}(k) > 0$.

36. Show that
$$\psi_n(x) := H_n(x)e^{-\frac{x^2}{2}} \in \mathcal{S}(\mathbb{R}),$$
    where $H_n(x)$ is the Hermite polynomial DLMF (12.7.2) of degree $n$ given by
$$H_n(x) := e^{\frac{x^2}{2}}\left(x - \frac{d}{dx}\right)^n e^{-\frac{x^2}{2}},$$
    are eigenfunctions of the Fourier transform: $\widehat{\psi}_n(p) = (-\mathrm{i})^n \psi_n(p)$.

37. Prove the Poisson summation formula
$$\sum_{n \in \mathbb{Z}} f(n)e^{-\mathrm{i}nx} = \sqrt{2\pi} \sum_{m \in \mathbb{Z}} \widehat{f}(x + 2\pi m),$$
    where $f \in L^1(\mathbb{R})$ satisfies $|f(x)| + |\widehat{f}(x)| \le \frac{C}{(1+|x|)^\alpha}$ for some $\alpha > 1$. (Hint: Compute the Fourier coefficients of the right-hand side. To this end observe that the integrals over $[-\pi, \pi]$ give a tiling of $\mathbb{R}$ when $m$ runs through all values in $\mathbb{Z}$.)

38. Suppose $f \in L^1(\mathbb{R})$ satisfies $|f(x)| + |\widehat{f}(x)| \le \frac{C}{(1+|x|)^\alpha}$ for some $\alpha > 1$. Prove the Whittaker–Shannon interpolation formula:
$$f(x) = \sum_{n \in \mathbb{Z}} f(n)\,\mathrm{sinc}(\pi(x - n))$$
    provided $\mathrm{supp}(\widehat{f}) \subseteq [-\pi, \pi]$. Here $\mathrm{sinc}(x) := \frac{\sin(x)}{x}$. (Hint: Use the Poisson summation formula to express $\widehat{f}$ and take the inverse Fourier transform.)

39. Show
$$\int_0^\infty \frac{\sin(x)^2}{x^2}dx = \frac{\pi}{2}.$$
    (Hint: Problem 35 (i).)

40. Suppose $f(x)e^{k|x|} \in L^1(\mathbb{R})$ for some $k > 0$. Then $\widehat{f}(p)$ has an analytic extension to the strip $|\mathrm{Im}(p)| < k$.

41. Show that
$$\|f\| := \inf\{\|f_0\|_{p_0} + \|f_1\|_{p_1}\,|\,f = f_0 + f_1,\ f_0 \in L^{p_0},\ f_1 \in L^{p_1}\}$$
    turns $L^{p_0} + L^{p_1}$ into a Banach space.

42. Use dilations $f(x) \mapsto f(\lambda x)$, $\lambda > 0$, to show that an inequality $\|\widehat{f}\|_q \le C_{p,q}\|f\|_p$ can only hold if $\frac{1}{p} + \frac{1}{q} = 1$.

43. Show that the Fourier transform does not extend to a continuous map $\mathcal{F} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, for $p > 2$. Use the closed graph theorem to conclude that $\mathcal{F}$ is not onto for $1 \le p < 2$. (Hint for the case $n = 1$: Consider $\phi_z(x) = \exp(-zx^2/2)$ for $z = \lambda + i\omega$ with $\lambda > 0$.)

44. Show that $E_f = 0$ if and only if $f = 0$. Moreover, show $E_{f+g}(r + s) \le E_f(r) + E_g(s)$ and $E_{\alpha f}(r) = E_f(r/|\alpha|)$ for $\alpha \ne 0$. Conclude that $L^{p,w}(X, d\mu)$ is a quasinormed space with

$$\|f + g\|_{p,w} \le 2\big(\|f\|_{p,w} + \|g\|_{p,w}\big), \qquad \|\alpha f\|_{p,w} = |\alpha|\|f\|_{p,w}.$$

45. Show $f(x) = |x|^{-n/p} \in L^{p,w}(\mathbb{R}^n)$. Compute $\|f\|_{p,w}$.

46. Compute the maximal function of $f(x) := \chi_{(-1,1)}(x)$.