UE Funktionalanalysis 1

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WS2009/10

1. Show that $|d(x,y) - d(z,y)| \le d(x,z)$.

2. Show the **quadrangle inequality** $|d(x,y) - d(x',y')| \le d(x,x') + d(y,y')$.

3. Let X be some space together with a sequence of distance functions d_n, $n \in \mathbb{N}$. Show that
$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x,y)}{1 + d_n(x,y)}$$
is again a distance function.

4. Show that the closure satisfies $\overline{\overline{U}} = \overline{U}$.

5. Let $U \subseteq V$ be subsets of a metric space X. Show that if U is dense in V and V is dense in X, then U is dense in X.

6. Show that any open set $O \subseteq \mathbb{R}$ can be written as a countable union of disjoint intervals. (Hint: Let $\{I_\alpha\}$ be the set of all maximal subintervals of O; that is, $I_\alpha \subseteq O$ and there is no other subinterval of O which contains I_α. Then this is a cover of disjoint intervals which has a countable subcover.)

7. Let X be a Banach space. Show that $\sum_{j=1}^{\infty} \|f_j\| < \infty$ implies that
$$\sum_{j=1}^{\infty} f_j = \lim_{n\to\infty} \sum_{j=1}^{n} f_j$$
exists. The series is called **absolutely convergent** in this case.

8. Show that $\ell^\infty(\mathbb{N})$ is a Banach space.

9. Show that $\ell^\infty(\mathbb{N})$ is not separable. (Hint: Consider sequences which take only the value one and zero. How many are there? What is the distance between two such sequences?)

10. Show that in a Hilbert space
$$\sum_{1\le j<k\le n} \|x_j - x_k\|^2 + \|\sum_{1\le j\le n} x_j\|^2 = n \sum_{1\le j\le n} \|x_j\|^2.$$

11. Show that the maximum norm on $C[0,1]$ does not satisfy the parallelogram law.

12. In a Banach space the unit ball is convex by the triangle inequality. A Banach space X is called **uniformly convex** if for every $\varepsilon > 0$ there is some δ such that $\|x\| \le 1$, $\|y\| \le 1$, and $\|\frac{x+y}{2}\| \ge 1 - \delta$ imply $\|x - y\| \le \varepsilon$.

Geometrically this implies that if the average of two vectors inside the closed unit ball is close to the boundary, then they must be close to each other.

Show that a Hilbert space is uniformly convex and that one can choose $\delta(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$. Draw the unit ball for $\mathbb{R}^2$ for the norms $\|x\|_1 = |x_1| + |x_2|$, $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2}$, and $\|x\|_\infty = \max(|x_1|, |x_2|)$. With which of these norms is $\mathbb{R}^2$ uniformly convex?

(Hint: For the first part use the parallelogram law.)

13. Consider $X = \mathbb{C}^n$ and let $A : X \to X$ be a matrix. Equip X with the norm (show that this is a norm)

$$\|x\|_\infty = \max_{1 \le j \le n} |x_j|$$

and compute the operator norm $\|A\|$ with respect to this matrix in terms of the matrix entries. Do the same with respect to the norm

$$\|x\|_1 = \sum_{1 \le j \le n} |x_j|.$$

14. Show that the integral operator

$$(Kf)(x) = \int_0^1 K(x,y) f(y) dy,$$

where $K(x,y) \in C([0,1] \times [0,1])$, defined on $\mathfrak{D}(K) = C[0,1]$ is a bounded operator both in $X = C[0,1]$ (max norm) and $X = \mathcal{L}^2_{cont}(0,1)$.

15. Show that the set of differentiable functions $C^1(I)$ becomes a Banach space if we set $\|f\|_{\infty,1} = \max_{x \in I} |f(x)| + \max_{x \in I} |f'(x)|$.

16. Show that $\|AB\| \le \|A\| \|B\|$ for every $A, B \in \mathfrak{L}(X)$. Conclude that the multiplication is continuous: $A_n \to A$ and $B_n \to B$ imply $A_n B_n \to AB$.

17. Let

$$f(z) = \sum_{j=0}^{\infty} f_j z^j, \qquad |z| < R,$$

be a convergent power series with convergence radius $R > 0$. Suppose A is a bounded operator with $\|A\| < R$. Show that

$$f(A) = \sum_{j=0}^{\infty} f_j A^j$$

exists and defines a bounded linear operator.

18. Let $\{u_j\}$ be some orthonormal basis. Show that a bounded linear operator A is uniquely determined by its matrix elements $A_{jk} = \langle u_j, A u_k \rangle$ with respect to this basis.

19. Show that an orthogonal projection $P_M \ne 0$ has norm one.

20. Suppose $P \in \mathfrak{L}(\mathfrak{H})$ satisfies
$$P^2 = P \qquad \text{and} \qquad \langle Pf, g\rangle = \langle f, Pg\rangle$$
and set $M = \mathrm{Ran}(P)$. Show
   - $Pf = f$ for $f \in M$ and M is closed,
   - $g \in M^\perp$ implies $Pg \in M^\perp$ and thus $Pg = 0$,

   and conclude $P = P_M$.
21. Let $\mathfrak{H}$ a Hilbert space and let $u, v \in \mathfrak{H}$. Show that the operator
$$Af = \langle u, f\rangle v$$
is bounded and compute its norm. Compute the adjoint of A.
22. Prove
$$\|A\| = \sup_{\|f\|=\|g\|=1} |\langle f, Ag\rangle|$$
(Hint: Use $\|f\| = \sup_{\|g\|=1} |\langle g, f\rangle|$.)
23. Show
$$\mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp.$$
24. Show that compact operators form an ideal.
25. Show that adjoint of the integral operator
$$(Kf)(x) = \int_a^b K(x,y) f(y) dy,$$
where $K(x,y) \in C([a,b] \times [a,b])$, defined on $\mathcal{L}^2_{cont}(a,b)$, is the integral operator with kernel $K(y,x)^*$.
26. Show that if A is bounded, then every eigenvalue α satisfies $|\alpha| \le \|A\|$.
27. Find the eigenvalues and eigenfunctions of the integral operator
$$(Kf)(x) = \int_0^1 u(x) v(y) f(y) dy$$
in $\mathcal{L}^2_{cont}(0,1)$, where $u(x)$ and $v(x)$ are some given continuous functions.
28. Find the eigenvalues and eigenfunctions of the integral operator
$$(Kf)(x) = 2\int_0^1 (2xy - x - y + 1) f(y) dy$$
in $\mathcal{L}^2_{cont}(0,1)$.
29. Show that the resolvent $R_A(z) = (A - z)^{-1}$ (provided it exists and is densely defined) of a symmetric operator A is again symmetric for $z \in \mathbb{R}$. (Hint: $g \in \mathfrak{D}(R_A(z))$ if and only if $g = (A - z)f$ for some $f \in \mathfrak{D}(A)$).

30. Show that $\mathrm{Ker}(A^*A) = \mathrm{Ker}(A)$ for any $A \in \mathfrak{L}(\mathfrak{H})$.

31. Compute $\mathrm{Ker}(1-K)$ and $\mathrm{Ran}(1-K)^\perp$ for the operator $K = \langle v, .\rangle u$, where $u, v \in \mathfrak{H}$ satisfy $\langle u, v\rangle = 1$.

32. Call two numbers $x, y \in \mathbb{R}/\mathbb{Z}$ equivalent if $x - y$ is rational. Construct the set V by choosing one representative from each equivalence class. Show that V cannot be measurable with respect to any nontrivial finite translation invariant measure on $\mathbb{R}/\mathbb{Z}$. (Hint: How can you build up $\mathbb{R}/\mathbb{Z}$ from translations of V?)

33. Show that the set $B(X)$ of bounded measurable functions with the sup norm is a Banach space. Show that the set $S(X)$ of simple functions is dense in $B(X)$. Show that the integral is a bounded linear functional on $B(X)$ if $\mu(X) < \infty$. (Hence BLT Theorem could be used to extend the integral from simple to bounded measurable functions.)

34. Show that the dominated convergence theorem implies (under the same assumptions)
$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

35. Let $X \subseteq \mathbb{R}$, Y be some measure space, and $f : X \times Y \to \mathbb{R}$. Suppose $y \mapsto f(x, y)$ is measurable for every x and $x \mapsto f(x, y)$ is continuous for every y. Show that
$$F(x) = \int_A f(x, y)\, d\mu(y)$$
is continuous if there is an integrable function $g(y)$ such that $|f(x, y)| \le g(y)$.

36. Let $X \subseteq \mathbb{R}$, Y be some measure space, and $f : X \times Y \to \mathbb{R}$. Suppose $y \mapsto f(x, y)$ is measurable for all x and $x \mapsto f(x, y)$ is differentiable for a.e. y. Show that
$$F(x) = \int_A f(x, y)\, d\mu(y)$$
is differentiable if there is an integrable function $g(y)$ such that $|\frac{\partial}{\partial x} f(x, y)| \le g(y)$. Moreover, $y \mapsto \frac{\partial}{\partial x} f(x, y)$ is measurable and
$$F'(x) = \int_A \frac{\partial}{\partial x} f(x, y)\, d\mu(y)$$
in this case.

37. Suppose $\mu(X) < \infty$. Show that $L^\infty(X, d\mu) \subseteq L^p(X, d\mu)$ and
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty, \qquad f \in L^\infty(X, d\mu).$$

38. Construct a function $f \in L^p(0, 1)$ which has a singularity at every rational number in $[0, 1]$ (such that the essential supremum is infinite on every open subinterval). (Hint: Start with the function $f_0(x) = |x|^{-\alpha}$ which has a single singularity at 0, then $f_j(x) = f_0(x - x_j)$ has a singularity at x_j.)

39. Show the following generalization of Hölder's inequality:

$$\|f\,g\|_r \le \|f\|_p \|g\|_q, \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

40. Show that

$$\|u\|_{p_0} \le \mu(X)^{\frac{1}{p_0} - \frac{1}{p}} \|u\|_p, \qquad 1 \le p_0 \le p.$$

(Hint: Hölder's inequality.)

41. Let $0 < \theta < 1$. Show that if $f \in L^{p_1} \cap L^{p_2}$, then $f \in L^p$ and

$$\|f\|_p \le \|f\|_{p_1}^{\theta} \|f\|_{p_2}^{1-\theta},$$

where $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$.

42. Let $\mathfrak{H} = \ell^2(\mathbb{N})$ and let A be multiplication by a sequence $a = (a_j)_{j=1}^{\infty}$. Show that A is Hilbert–Schmidt if and only if $a \in \ell^2(\mathbb{N})$. Furthermore, show that $\|A\|_{HS} = \|a\|$ in this case.

43. Show that $K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $f_n \mapsto \sum_{j\in\mathbb{N}} k_{n+j} f_j$ is Hilbert–Schmidt with $\|K\|_{HS} \le \|c\|_1$ if $|k_j| \le c_j$, where c_j is decreasing and summable.

44. Suppose $A : \mathfrak{D}(A) \to \operatorname{Ran}(A)$ is closed and injective. Show that A^{-1} defined on $\mathfrak{D}(A^{-1}) = \operatorname{Ran}(A)$ is closed as well.

Conclude that in this case $\operatorname{Ran}(A)$ is closed if and only if A^{-1} is bounded.

45. Show that the differential operator $A = \frac{d}{dx}$ defined on $\mathfrak{D}(A) = C^1[0,1] \subset C[0,1]$ (sup norm) is a closed operator.

46. Let X be some Banach space. Show that

$$\|x\| = \sup_{\ell \in X^*,\, \|\ell\|=1} |\ell(x)|$$

for all $x \in X$.

47. Show that $\|l_y\| = \|y\|_q$, where $l_y \in \ell^p(\mathbb{N})^*$ is given by

$$l_y(x) = \sum_{n\in\mathbb{N}} y_n x_n.$$

(Hint: Choose $x \in \ell^p$ such that $x_n y_n = |y_n|^q$.)

48. Show that every $l \in \ell^p(\mathbb{N})^*$, $1 \le p < \infty$, can be written as

$$l(x) = \sum_{n\in\mathbb{N}} y_n x_n$$

with some $y \in \ell^q(\mathbb{N})$. (Hint: To see $y \in \ell^q(\mathbb{N})$ consider x^N defined such that $x_n y_n = |y_n|^q$ for $n \le N$ and $x_n = 0$ for $n > N$. Now look at $|\ell(x^N)| \le \|\ell\| \|x^N\|_p$.)

49. Let $c_0(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$ be the subspace of sequences which converge to 0, and $c(\mathbb{N}) \subset \ell^\infty(\mathbb{N})$ the subspace of convergent sequences.

(a) Show that $c_0(\mathbb{N})$, $c(\mathbb{N})$ are both Banach spaces and that $c(\mathbb{N}) = \text{span}\{c_0(\mathbb{N}), e\}$, where $e = (1, 1, 1, \dots) \in c(\mathbb{N})$.

(b) Show that every $l \in c_0(\mathbb{N})^*$ can be written as

$$l(x) = \sum_{n\in\mathbb{N}} y_n x_n$$

with some $y \in \ell^1(\mathbb{N})$ which satisfies $\|y\|_1 = \|\ell\|$.

(c) Show that every $l \in c(\mathbb{N})^*$ can be written as

$$l(x) = \sum_{n\in\mathbb{N}} y_n x_n + y_0 \lim_{n\to\infty} x_n$$

with some $y \in \ell^1(\mathbb{N})$ which satisfies $|y_0| + \|y\|_1 = \|\ell\|$.

50. Suppose $\ell_n \to \ell$ in X^* and $x_n \rightharpoonup x$ in X. Then $\ell_n(x_n) \to \ell(x)$.

51. Show that $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$ for $A \in \mathfrak{L}(X)$.

52. Show that if $\{\ell_j\} \subseteq X^*$ is some total set, then $x_n \rightharpoonup x$ if and only if x_n is bounded and $\ell_j(x_n) \to \ell_j(x)$ for all j. Show that this is wrong without the boundedness assumption (Hint: Take e.g. $X = \ell^2(\mathbb{N})$).

53. Show that for $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, the convolution

$$(g * f)(x) = \int_{\mathbb{R}^n} g(x-y)f(y)dy = \int_{\mathbb{R}^n} g(y)f(x-y)dy$$

is in $L^p(\mathbb{R}^n)$ and satisfies Young's inequality

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

(Hint: Without restriction $\|f\|_1 = 1$. Now use Jensen and Fubini.)

54. Show that the multiplication in a Banach algebra X is continuous: $x_n \to x$ and $y_n \to y$ imply $x_n y_n \to xy$.

55. Show that $L^1(\mathbb{R}^n)$ with convolution as multiplication is a commutative Banach algebra without identity.

56. Show that $\sigma(x) \subset \{t \in \mathbb{R} | t \ge 0\}$ if x is positive.