UE Funktionalanalysis

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Note: References refer to the lecture notes.

1. Suppose $\sum_{n=1}^{\infty} |c_n| < \infty$. Show that
$$u(t,x) := \sum_{n=1}^{\infty} c_n \mathrm{e}^{-(\pi n)^2 t} \sin(n\pi x),$$
is continuous for $(t,x) \in [0,\infty) \times [0,1]$ and solves the heat equation for $(t,x) \in (0,\infty) \times [0,1]$. (Hint: Weierstrass M-test. When can you interchange the order of summation and differentiation?)

2. Show that $|\|f\| - \|g\|| \le \|f - g\|$.

3. Let X be a Banach space. Show that the norm, vector addition, and multiplication by scalars are continuous. That is, if $f_n \to f$, $g_n \to g$, and $\alpha_n \to \alpha$, then $\|f_n\| \to \|f\|$, $f_n + g_n \to f + g$, and $\alpha_n g_n \to \alpha g$.

4. Prove Young's inequality
$$\alpha^{1/p}\beta^{1/q} \le \frac{1}{p}\alpha + \frac{1}{q}\beta, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha, \beta \ge 0.$$
Show that equality occurs precisely if $\alpha = \beta$. (Hint: Take logarithms on both sides.)

5. Show that $\ell^p(\mathbb{N})$, $1 \le p < \infty$, is complete.

6. Show that there is equality in the Hölder inequality for $1 < p < \infty$ if and only if either $a = 0$ or $|b_j|^q = \alpha |a_j|^p$ for all $j \in \mathbb{N}$. Show that we have equality in the triangle inequality for $\ell^1(\mathbb{N})$ if and only if $a_j b_j^* \ge 0$ for all $j \in \mathbb{N}$ (here the '$*$' denotes complex conjugation). Show that we have equality in the triangle inequality for $\ell^p(\mathbb{N})$ with $1 < p < \infty$ if and only if $a = 0$ or $b = \alpha a$ with $\alpha \ge 0$.

7. Let X be a normed space. Show that the following conditions are equivalent.

   (i) If $\|x + y\| = \|x\| + \|y\|$ then $y = \alpha x$ for some $\alpha \ge 0$ or $x = 0$.
   (ii) If $\|x\| = \|y\| = 1$ and $x \ne y$ then $\|\lambda x + (1-\lambda) y\| < 1$ for all $0 < \lambda < 1$.
   (iii) If $\|x\| = \|y\| = 1$ and $x \ne y$ then $\frac{1}{2}\|x + y\| < 1$.
   (iv) The function $x \mapsto \|x\|^2$ is strictly convex.

   A norm satisfying one of them is called strictly convex.

   Show that $\ell^p(\mathbb{N})$ is strictly convex for $1 < p < \infty$ but not for $p = 1, \infty$.

8. Show that $p_0 \le p$ implies $\ell^{p_0}(\mathbb{N}) \subset \ell^p(\mathbb{N})$ and $\|a\|_p \le \|a\|_{p_0}$. Moreover, show
$$\lim_{p\to\infty} \|a\|_p = \|a\|_\infty.$$

9. Show that $\ell^\infty(\mathbb{N})$ is not separable. (Hint: Consider sequences which take only the value one and zero. How many are there? What is the distance between two such sequences?)

10. Formally extend the definition of $\ell^p(\mathbb{N})$ to $p \in (0,1)$. Show that $\|.\|_p$ does not satisfy the triangle inequality. However, show that it is a quasinormed space, that is, it satisfies all requirements for a normed space except for the triangle inequality which is replaced by
$$\|a+b\| \le K(\|a\| + \|b\|)$$
with some constant $K \ge 1$. Show, in fact,
$$\|a+b\|_p \le 2^{1/p-1}(\|a\|_p + \|b\|_p), \qquad p \in (0,1).$$
Moreover, show that $\|.\|_p^p$ satisfies the triangle inequality in this case, but of course it is no longer homogeneous (but at least you can get an honest metric $d(a,b) = \|a-b\|_p^p$ which gives rise to the same topology). (Hint: Show $\alpha + \beta \le (\alpha^p + \beta^p)^{1/p} \le 2^{1/p-1}(\alpha+\beta)$ for $0 < p < 1$ and $\alpha, \beta \ge 0$.)

11. Let I be a compact interval and consider $X = C(I)$. Which of following sets are subspaces of X? If yes, are they closed?
    (i) monotone functions
    (ii) even functions
    (iii) continuous piecewise linear functions

12. Let I be a compact interval. Show that the set $Y := \{f \in C(I) | f(x) > 0\}$ is open in $X := C(I)$. Compute its closure.

13. Which of the following bilinear forms are scalar products on $\mathbb{R}^n$?
    (i) $s(x,y) := \sum_{j=1}^n (x_j + y_j)$.
    (ii) $s(x,y) := \sum_{j=1}^n \alpha_j x_j y_j$, $\alpha \in \mathbb{R}^n$.

14. Show that the maximum norm on $C[0,1]$ does not satisfy the parallelogram law.

15. Suppose $\mathfrak{Q}$ is a complex vector space. Let $s(f,g)$ be a sesquilinear form on $\mathfrak{Q}$ and $q(f) := s(f,f)$ the associated quadratic form. Prove the **parallelogram law**
$$q(f+g) + q(f-g) = 2q(f) + 2q(g)$$
and the **polarization identity**
$$s(f,g) = \frac{1}{4}\left(q(f+g) - q(f-g) + \mathrm{i}\, q(f-\mathrm{i}g) - \mathrm{i}\, q(f+\mathrm{i}g)\right).$$
Show that $s(f,g)$ is symmetric if and only if $q(f)$ is real-valued.

    Note, that if $\mathfrak{Q}$ is a real vector space, then the parallelogram law is unchanged but the polarization identity in the form $s(f,g) = \frac{1}{4}(q(f+g) - q(f-g))$ will only hold if $s(f,g)$ is symmetric.

16. Provide a detailed proof of Theorem 1.10.

17. Show that a subset $\mathcal{K} \subset c_0(\mathbb{N})$ is relatively compact if and only if there is a nonnegative sequence $a \in c_0(\mathbb{N})$ such that $|b_n| \le a_n$ for all $n \in \mathbb{N}$ and all $b \in \mathcal{K}$.

18. Which of the following families are relatively compact in $C[0,1]$?
    (i) $F = \{f \in C^1[0,1] | \, \|f\|_\infty \le 1\}$
    (ii) $F = \{f \in C^1[0,1] | \, \|f'\|_\infty \le 1\}$
    (iii) $F = \{f \in C^1[0,1] | \, \|f\|_\infty \le 1, \|f'\|_2 \le 1\}$

19. Let $X := C[0,1]$. Show that $\ell(f) := \int_0^1 f(x)dx$ is a linear functional. Compute its norm. Is the norm attained? What if we replace X by $X_0 := \{f \in C[0,1] | f(0) = 0\}$ (in particular, check that this is a closed subspace)?

20. Show that the integral operator

$$(Kf)(x) := \int_0^1 K(x,y)f(y)dy,$$

where $K(x,y) \in C([0,1]\times[0,1])$, defined on $\mathfrak{D}(K) := C[0,1]$, is a bounded operator in $X := \mathcal{L}^2_{cont}(0,1)$.

21. Let I be a compact interval. Show that the set of differentiable functions $C^1(I)$ becomes a Banach space if we set $\|f\|_{\infty,1} := \max_{x\in I} |f(x)| + \max_{x\in I} |f'(x)|$.

22. Suppose $B \in \mathfrak{L}(X)$ with $\|B\| < 1$. Then $\mathbb{I} + B$ is invertible with

$$(\mathbb{I} + B)^{-1} = \sum_{n=0}^{\infty} (-1)^n B^n.$$

Consequently for $A, B \in \mathfrak{L}(X,Y)$, $A+B$ is invertible if A is invertible and $\|B\| < \|A^{-1}\|^{-1}$.

23. Let X_j, $j = 1, \dots, n$, be Banach spaces. Let $X := \bigoplus_{p,j=1}^n X_j$ be the Cartesian product $X_1 \times \cdots \times X_n$ together with the norm

$$\|(x_1,\dots,x_n)\|_p := \begin{cases} \left(\sum_{j=1}^n \|x_j\|^p\right)^{1/p}, & 1 \le p < \infty, \\ \max_{j=1,\dots,n} \|x_j\|, & p = \infty. \end{cases}$$

Show that X is a Banach space. Show that all norms are equivalent and that this sum is associative $(X_1 \oplus_p X_2) \oplus_p X_3 = X_1 \oplus_p (X_2 \oplus_p X_3)$.

24. Compute $\|[e]\|$ in $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$, where $e := (1,1,1,\dots)$.

25. Suppose $A \in \mathfrak{L}(X,Y)$. Show that $\mathrm{Ker}(A)$ is closed. Suppose $M \subseteq \mathrm{Ker}(A)$ is a closed subspace. Show that the induced map $\tilde{A} : X/M \to Y$, $[x] \mapsto Ax$ is a well-defined operator satisfying $\|\tilde{A}\| = \|A\|$ and $\mathrm{Ker}(\tilde{A}) = \mathrm{Ker}(A)/M$. In particular, $\tilde{A}$ is injective for $M = \mathrm{Ker}(A)$.

26. Given some vectors $f_1, \dots, f_n$ we define their Gram determinant as

$$\Gamma(f_1, \dots, f_n) := \det\left(\langle f_j, f_k\rangle\right)_{1\le j,k\le n}.$$

Show that the Gram determinant is nonzero if and only if the vectors are linearly independent. Moreover, show that in this case

$$\operatorname{dist}(g, \operatorname{span}\{f_1, \dots, f_n\})^2 = \frac{\Gamma(f_1, \dots, f_n, g)}{\Gamma(f_1, \dots, f_n)}$$

and

$$\Gamma(f_1, \dots, f_n) \le \prod_{j=1}^n \|f_j\|^2.$$

with equality if the vectors are orthogonal. (Hint: First establish $\Gamma(f_1, \dots, f_j + \alpha f_k, \dots, f_n) = \Gamma(f_1, \dots, f_n)$ for $j \ne k$ and use it to investigate how Γ changes when you apply the Gram–Schmidt procedure?)

27. Show that $\ell(a) = \sum_{j=1}^\infty \frac{a_j + a_{j+2}}{2^j}$ defines a bounded linera functional on $X := \ell^2(\mathbb{N})$. Compute its norm.

28. Suppose $P \in \mathfrak{L}(\mathfrak{H})$ satisfies

$$P^2 = P \qquad \text{and} \qquad \langle Pf, g\rangle = \langle f, Pg\rangle$$

and set $M := \operatorname{Ran}(P)$. Show

- $Pf = f$ for $f \in M$ and M is closed,
- $\operatorname{Ker}(P) = M^\perp$

and conclude $P = P_M$.

29. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be Hilbert spaces and let $u \in \mathfrak{H}_1$, $v \in \mathfrak{H}_2$. Show that the operator

$$Af := \langle u, f\rangle v$$

is bounded and compute its norm. Compute the adjoint of A.

30. Prove

$$\|A\| = \sup_{\|g\|_{\mathfrak{H}_2} = \|f\|_{\mathfrak{H}_1} = 1} |\langle g, Af\rangle_{\mathfrak{H}_2}| \le C.$$

(Hint: Use $\|f\| = \sup_{\|g\|=1} |\langle g, f\rangle|$ — compare Theorem 1.5.)

31. Suppose $A \in \mathfrak{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ has a bounded inverse $A^{-1} \in \mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$. Show $(A^{-1})^* = (A^*)^{-1}$.

32. Show

$$\operatorname{Ker}(A^*) = \operatorname{Ran}(A)^\perp.$$

33. Show that $f \otimes \tilde{f} = 0$ if and only if $f = 0$ or $\tilde{f} = 0$.

34. Show Theorem 3.1.

35. Is the left shift $(a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, \dots)$ compact in $\ell^2(\mathbb{N})$?

36. Is the operator $\frac{d}{dx} : C^k[0,1] \to C[0,1]$ compact for $k = 1,2$? (Hint: Problem 18 and Example 3.3 from the lecture notes.)

37. Let $\mathfrak{H} := \mathcal{L}^2_{cont}(0,1)$. Find the eigenvalues and eigenfunctions of the differentiation operator $A : \mathfrak{D}(A) \subseteq \mathfrak{H} \to \mathfrak{H}$, $f(x) \mapsto f'(x)$ for the following domains

    (i) $\mathfrak{D}(A) := C^1[0,1]$.

    (ii) $\mathfrak{D}(A) := \{f \in C^1[0,1] | f(0) = 0\}$.

    (iii) $\mathfrak{D}(A) := \{f \in C^1[0,1] | f(0) = f(1)\}$.

38. Find the eigenvalues and eigenfunctions of the integral operator $K \in \mathfrak{L}(\mathcal{L}^2_{cont}(0,1))$ given by

$$(Kf)(x) := \int_0^1 u(x)v(y)f(y)dy,$$

where $u, v \in C([0,1])$ are some given continuous functions.

39. Find the eigenvalues and eigenfunctions of the integral operator $K \in \mathfrak{L}(\mathcal{L}^2_{cont}(0,1))$ given by

$$(Kf)(x) := 2\int_0^1 (2xy - x - y + 1)f(y)dy.$$

40. Let $\mathfrak{H} := \mathcal{L}^2_{cont}(0,1)$. Show that the Volterra integral operator $K : \mathfrak{H} \to \mathfrak{H}$ defined by

$$(Kf)(x) := \int_a^x K(x,y)f(y)dy,$$

where $K(x,y) \in C([a,b] \times [a,b])$, has no eigenvalues except for 0. Show that 0 is no eigenvalue if $K(x,y)$ is C^1 and satisfies $K(x,x) > 0$. Why does this not contradict Theorem 3.6? (Hint: Gronwall's inequality.)

41. Show that the resolvent $R_A(z) = (A - z)^{-1}$ (provided it exists and is densely defined) of a symmetric operator A is again symmetric for $z \in \mathbb{R}$. (Hint: $g \in \mathfrak{D}(R_A(z))$ if and only if $g = (A - z)f$ for some $f \in \mathfrak{D}(A)$.)

42. Show that for our Sturm–Liouville operator $u_\pm(z,x)^* = u_\pm(z^*,x)$. (Hint: Which differential equation does $u_\pm(z,x)^*$ solve?)

43. Show that

$$\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(Hint: Use the trace formula (3.29).)

44. Consider the Sturm–Liouville problem on a compact interval $[a,b]$ with domain

$$\mathfrak{D}(L) = \{f \in C^2[a,b] | f'(a) = f'(b) = 0\}.$$

Show that Theorem 3.11 continues to hold.

45. Every subset of a meager set is again meager.

46. Let X be the space of sequences with finitely many nonzero terms together with the sup norm. Consider the family of operators $\{A_n\}_{n\in\mathbb{N}}$ given by $(A_n a)_j := j a_j$, $j \le n$ and $(A_n a)_j := 0$, $j > n$. Then this family is pointwise bounded but not uniformly bounded. Does this contradict the Banach–Steinhaus theorem?

47. Show that a bilinear map $B : X \times Y \to Z$ is bounded, $\|B(x,y)\| \le C\|x\|\|y\|$, if and only if it is separately continuous with respect to both arguments. (Hint: Uniform boundedness principle.)

48. Show that a compact symmetric operator in an infinite-dimensional Hilbert space cannot be surjective.

49. Let $X := \mathbb{C}^3$ equipped with the norm $|(x,y,z)|_1 := |x|+|y|+|z|$ and $Y := \{(x,y,z)|x+y=0,\ z=0\}$. Find at least two extensions of $\ell(x,y,z) := x$ from Y to X which preserve the norm. What if we take $Y := \{(x,y,z)|x+y=0\}$?

50. Consider $X := C[0,1]$ and let $f_0(x) := 1-2x$. Find at least two linear functional with minimal norm such that $\ell(f_0) = 1$.

51. Show that the extension from Corollary 4.11 is unique if X^* is strictly convex. (Hint: Problem 7.)

52. Let X be some normed space. Show that

$$\|x\| = \sup_{\ell\in V,\, \|\ell\|=1} |\ell(x)|,$$

where $V \subset X^*$ is some dense subspace. Show that equality is attained if $V = X^*$.

53. Suppose M_1, M_2 are closed subspaces of X. Show

$$M_1 \cap M_2 = (M_1^\perp + M_2^\perp)_\perp, \qquad M_1^\perp \cap M_2^\perp = (M_1 + M_2)^\perp$$

and

$$(M_1 \cap M_2)^\perp \supseteq \overline{(M_1^\perp + M_2^\perp)}, \qquad (M_1^\perp \cap M_2^\perp)_\perp = \overline{(M_1 + M_2)}.$$

54. Show that if $A \in \mathfrak{L}(X,Y)$, then $\operatorname{Ran}(A)^\perp = \operatorname{Ker}(A')$ and $\operatorname{Ran}(A')_\perp = \operatorname{Ker}(A)$.

55. Suppose $\ell_n \to \ell$ in X^* and $x_n \rightharpoonup x$ in X. Then $\ell_n(x_n) \to \ell(x)$. Similarly, suppose s-lim $\ell_n = \ell$ and $x_n \to x$. Then $\ell_n(x_n) \to \ell(x)$. Does this still hold if s-lim $\ell_n = \ell$ and $x_n \rightharpoonup x$?

56. Establish Lemma 4.34 in the case of weak convergence. (Hint: The formula

$$\|A\| = \sup_{x\in X,\, \|x\|=1;\, \ell\in V,\, \|\ell\|=1} |\ell(Ax)|,$$

might be useful.)