Proseminar Advanced Complex Analysis

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Note: Most problems are taken from W. Schlag, *A Course in Complex Analysis and Riemann Surfaces*, GSM 154, AMS, Providence, 2014.

1. Find a Möbius transformation that
   (i) takes $\{z \in \mathbb{C} : |z - 1 + \mathrm{i}| < 1\}$ onto $\{z \in \mathbb{C} : |z| < 1\}$.
   (ii) takes $\{z \in \mathbb{C} : \mathrm{Im}(z) > 1\}$ onto $\{z \in \mathbb{C} : |z| > 1\}$.

2. Discuss the mapping properties of $z \mapsto w = \frac{1}{2}(z + \frac{1}{z})$ on $\{z \in \mathbb{C} : |z| < 1\}$. Is it one-to-one there? What is the image of $\{z \in \mathbb{C} : |z| < 1\}$ in the w-plane? What happens on $\{z \in \mathbb{C} : |z| = 1\}$ and $\{z \in \mathbb{C} : |z| > 1\}$? What is the image of the circles $\{z \in \mathbb{C} : |z| = r\}$ with $r < 1$, and of the ray $\{z \in \mathbb{C} : \mathrm{Arg}(z) = 0\}$ emanating from zero?

3. Let $T(z) = \frac{az+b}{cz+d}$ be a Mobius transformation.
   (i) Show that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ if and only if we can choose $a, b, c, d \in \mathbb{R}$
   (ii) Find all T such that $T(\mathbb{T}) = \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle.
   (iii) Find all T for which $T(\mathbb{D}) = \mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk.

4. (Schwarz lemma) Let $f \in \mathcal{H}(\mathbb{D})$ with $|f(z)| < 1$ for all $z \in \mathbb{D}$.
   Without any assumption on $f(0)$, prove that
   $$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \le \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|, \quad \forall z_1, z_2 \in \mathbb{D} \tag{1}$$
   and
   $$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}, \quad \forall z \in \mathbb{D} \tag{2}$$
   Show that equality in (1) for some pair $z_1 \neq z_2$ or in (2) for some $z \in \mathbb{D}$ implies that $f(z)$ is a fractional linear transformation.

5. Let $f \in \mathcal{H}(\mathbb{C})$ and $\mathrm{Re}(f(z)) > 0$. Show that f is constant.

6. Find the holomorphic function $f(z) = f(x + iy)$ with real part
   $$\frac{x(1 + x^2 + y^2)}{1 + 2x^2 - 2y^2 + (x^2 + y^2)^2}$$
   such that $f(0) = 0$.

7. Let $f \in \mathcal{H}(\mathbb{D})$ be such that $\mathrm{Re}(f(z)) > 0$ for all $z \in \mathbb{D}$, and $f(0) = a > 0$. Prove that $|f'(0)| \leq 2a$. Is this inequality sharp? If so, which functions attain it?

8. Give another, more elementary, proof of the fundamental theorem of algebra; see (Proposition 1.23) following these lines: Let $p(z)$ be a nonconstant polynomial. Show that $|p(z)|$ attains a minimum in the complex plane, say at z_0. If the polynomial $q(z) = p(z + z_0)$ starts with a nonzero constant term, obtain a contradiction by showing that we may find a small z such that $q(z)$ is closer to the origin than $q(0)$.

9. Find the Laurent expansion of the function $f(z) = \frac{1}{z(z-1)}$ in

   (i) $\{z \in \mathbb{C} | 0 < |z| < 1\}$

   (ii) $\{z \in \mathbb{C} | 1 < |z|\}$

10. Find all singularities of the following functions (state types of these singularities as well as order of poles)

    (i) $\frac{1}{z^2-1} \cos(\frac{\pi z}{z+1})$

    (ii) $\frac{1-\cos(z)}{\sin^2(z)}$

11. Find the principal part of the function $f(z) = \frac{\cos(z)}{(z-1)^2}$ at $z = 1$.

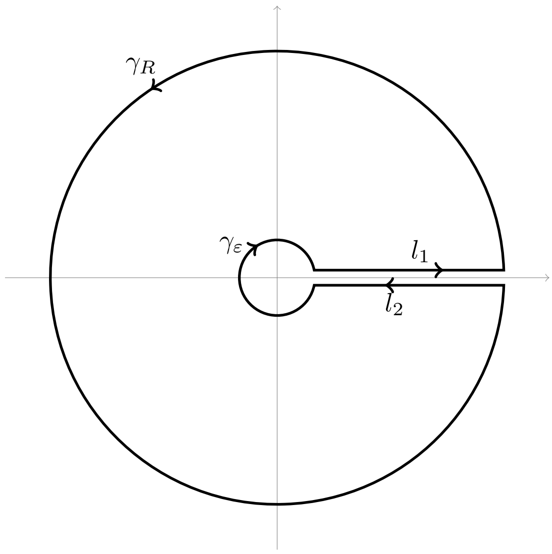
12. Compute the integral
$$\int_{\partial D} \frac{1}{1+z^4} dz,$$
where $D = \{z \in \mathbb{C} : |z - 1| < 1\}$.

13. Compute the integral
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2} dx$$

14. Compute the integral
$$\int_{-\pi}^{\pi} \frac{1}{5 + 3\cos\phi} d\phi$$

15. Prove Jordan's lemma:
Let $g(z) \in C(\overline{\{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}})$, $C_R = \{z = Re^{i\phi} : \phi \in [0, \pi]\}$ and $a > 0$. Then
$$\left| \int_{C_R} g(z) e^{iaz} dz \right| \leq M_R \frac{\pi}{a},$$
where $M_R = \max_{\phi \in [0,\pi]} |g(Re^{i\phi})|$.

16. Compute the integral
$$\int_0^{\infty} \frac{\sqrt{x}}{1+x^2} dx$$
(Hint: Use the following contour depicted below.)



17. Find the number of roots of:

    (i) $f(z) = z^4 - 3z + 1$ in $\{z \in \mathbb{C} | |z| < 1\}$
    (ii) $f(z) = z^6 - 6z + 10$ in $\{z \in \mathbb{C} | |z| > 1\}$

18. Show that the equation $ze^{\lambda - z} = 1$ with $\lambda > 1$ has exactly one zero in $\{z \in \mathbb{C} |\, |z| \leq 1\}$. Show that it is real and positive.

19. Show that the equation $\lambda - e^{-z} - z = 0$ with $\lambda > 1$ has exactly one zero in $\{z \in \mathbb{C} | \mathrm{Re} z \geq 0\}$.

20. Prove the Schwarz reflection principle.

    (i) Let Ω be an open set in the closed upper half-plane $\overline{\mathbb{H}}$ and denote $\Omega_0 = \Omega \cap \mathbb{H}$. Suppose $f \in \mathcal{H}(\Omega_0) \cap C(\Omega)$ with $\mathrm{Im} f(z) = 0$ for all $z \in \Omega \cap \partial\mathbb{H}$. Define

    $$F(z) := \begin{cases} f(z), & z \in \Omega \\ \overline{f(\overline{z})}, & z \in \Omega^-, \end{cases}$$

    where $\Omega^- = \{z : \overline{z} \in \Omega\}$. Prove that $F \in \mathcal{H}(\Omega \cup \Omega^-)$.
    (ii) Suppose $f \in \mathcal{H}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ so that $|f(z)| = 1$ on $|z| = 1$. If f does not vanish anywhere in $\mathbb{D}$, then prove that f is constant.

21. Find the image of $\{z \in \mathbb{C} | -\pi < \mathrm{Im}(z) < 0\}$ under the map $z \mapsto e^z$.

22. Find the image of $\{z \in \mathbb{C} | -\pi < \mathrm{Im}(z) < \pi,\ \mathrm{Re}(z) > 0\}$ under the map $z \mapsto \sinh(z)$.

23. Is there a bi-holomorphic map between $\{0 < |z| < 1\}$ and $\{\frac{1}{2} < |z| < 1\}$?

    (Hint: Notice that if such map existed, it would have an isolated singularity at 0. Which type of singularity could it be?)

24. Consider the power series

$$f(z) := \sum_{n=0}^{\infty} z^{2^n}, \quad |z| < 1$$

with radius of convergence 1. Prove that f is singular at every point of $\partial\mathbb{D}$.

(Hint: Let $\phi = \frac{2\pi l}{2^k}$, where $k, l \in \mathbb{N}$. Show that $|f(r\mathrm{e}^{\mathrm{i}\phi})| \to \infty$ as $r \to 1-$.)

25. Let $K_1 = D_1(4)$, $K_2 = D_1(4\mathrm{i})$, $K_3 = D_1(-4)$, $K_4 = D_1(-4\mathrm{i})$, where $D_1(a)$ is a closed disk of radius 1 and with center a. Show that there exists a sequence of entire functions f_n such that $f_n \to j$ uniformly on K_j for $j = 1, 2, 3, 4$.

26. Prove that there exists a sequence of polynomials p_n such that $p_n \to 1$ uniformly on compact subsets of $\{z \in \mathbb{C} | \mathrm{Re}(z) > 0\}$, $p_n \to -1$ uniformly on compact subsets of $\{z \in \mathbb{C} | \mathrm{Re}(z) < 0\}$ and $p_n \to 0$ uniformly on compact subsets of $\mathrm{i}\mathbb{R}$.

27. Prove that there exists a sequence of entire functions f_n such that $f_n \to 1$ uniformly on compact subsets of the open upper half-plane and (f_n) does not converge at any point of the open lower half-plane.

28. Show that a nonnegative harmonic function on $\mathbb{R}^2$ is constant.

29. Find all harmonic functions u such that $u(x, y) \le x^2 - y^2$.

30. Find all harmonic functions u such that $u_x(x, y) < u_y(x, y)$.

31. (Phragmen-Lindelöf principle) Let $\lambda \ge 1$ and let S be the sector

$$S := \{re^{i\theta} | 0 < r < \infty, |\theta| < \frac{\pi}{2\lambda}\}.$$

Let u be subharmonic on S and continuous up to the boundary $u \in C(\overline{S})$, and satisfy $u \le M$ on ∂S and $u(z) \le |z|^\rho$ in S where $\rho < \lambda$. Prove that $u \le M$ on S.

(Hint: Note that $v(z) = \mathrm{Re}(z^\rho)$ is harmonic. Using $v(z)$, introduce a family function f_ϵ such that (i) $f_\epsilon \le M$ on the boundary of some appropriately chosen bounded subdomain of S, (ii) f_ϵ decays at infinity in such a way that $f_\epsilon \le M$ on the complement of this subdomain, and (iii) $f_\epsilon \to u$ as $\epsilon \to 0$.)

32. (Hadamard's three lines lemma) Suppose f is holomorphic and bounded on a vertical strip $a \le \mathrm{Re} z \le b$. Show that the logarithm of $M(x) = \sup_y |f(x + iy)|$ is a convex function on $[a, b]$ (i.e. $M(x) \le M(a)^t M(b)^{1-t}$ for $x = ta + (1 - t)b$ with $0 \le t \le 1$).

33. Express the function $f(z) = e^z - 1$ as a product.

34. Find an atlas for the Riemann surface of $\sqrt{z^2 - 1}$ and conclude that it is equivalent to the Riemann sphere. (Hint: Show how $\mathbb{C}_\infty \setminus [-1, 1]$ can be mapped onto the unit disc – see Problem 2. In particular, verify that

this is continuous at ∞. Note that using the other *sign*, you get a map to the exterior of the unit disc. Now as a set the Riemann surface is $M = \{(z,\zeta)|z \in \mathbb{C}, \zeta^2 = z^2 - 1\} \cup \{(\infty, \pm\infty)\}$ – why do we need two points at ∞? Remove one point and map (bijective) M to $\mathbb{C}$ (you have two options). Compute the transition function.)