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Spectral Theory for Schrödinger Operators on Regular Tree Graphs

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SPECTRAL THEORY FOR SCHRÖDINGER OPERATORS
ON REGULAR TREE GRAPHS

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ABSTRACT

The presented work offers an introduction to the theory of regular tree graphs and the associated Schrödinger Operators.

We will start with a brief introduction, which recalls the required facts about Sturm-Liouville Operators and what the spectral theorem tells us about this operators. If you are not so comfortable with this terms, you should take a closer look at [12], which presents an more detailed introduction to these terms.

The second chapter generalizes a result from Barry Simon [9] on the connection between the absolutely continuous spectrum and boundedness of eigenfunctions to arbitrary Sturm-Liouville operators.

In the third chapter, we will investigate regular trees and Laplace operators on these trees. We introduce special subspaces, which reduce the Laplacian. Then, we will say how we can describe the spectrum with the help of these reducing subspaces. Finally, we will characterize the trees in relation to spectral characteristics with the help of the result derived in Chapter 2.

In the last chapter, we will show how the results of the previous chapters can be used to investigate the spectra of regular trees with constant branching number and constant edge length.

Thanks

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Contents

Chapter 1

Introduction

The following two sections present a brief introduction in Sturm–Liouville operators and the spectral theorem for these operators. The facts are mainly taken from [12].

1.1 Sturm-Liouville-Operator

The Sturm-Liouville operator is defined as follows:

$$(\tau u)(x) = \frac{1}{r(x)}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right) + V(x)u(x)\right), \quad u, pu' \in AC_{loc}(I) \tag{1.1}$$

in the Hilbert space

$$L^2((0,b), r(x)dx), \quad \langle u, v\rangle := \int_0^b \overline{u(x)}v(x)r(x)\mathrm{d}x, \tag{1.2}$$

where $I = (0, b) \in \mathbb{R}$ is an arbitrary open interval.
We require the following:

- $p^{-1} \in L^1_{loc}(I)$, positive
- $q \in L^1_{loc}(I)$, real-valued
- $r \in L^1_{loc}(I)$, positive

The maximal domain of definition for τ in $L^2(I, r(x)\mathrm{d}x)$ is given by

$$\mathfrak{D}(\tau) = \left\{u \in L^2(I, r(x)\mathrm{d}x) : u, pu' \in AC_{loc}(I), \tau u \in L^2(I, r(x)\mathrm{d}x)\right\}. \tag{1.3}$$

First of all we search for a space, where τ is symmetric. The modified Wronskian associated with this problem is defined by:

$$W_x(u, v) = (p(uv' - u'v))(x). \tag{1.4}$$

It is straightforward to check, that the Wronskian of two solutions of $\tau u = zu$ is constant

$$W_x(u_1, u_2) = W(u_1, u_2) \text{ for } \tau u_{1,2} = zu_{1,2}$$

and nonzero if and only if u_1 and u_2 are linearly independent. If we have $u, v \in \mathfrak{D}(\tau)$, we get the following equation

$$\langle v, \tau u\rangle = W_b(\overline{v}, u) - W_a(\overline{v}, u) + \langle \tau v, u\rangle \,. \tag{1.5}$$

So we need, that the Wronskian of two such functions vanishes to render the operator τ symmetric. Therefore we set $A_0 u = \tau u$ and $\mathfrak{D}(A_0) = \mathfrak{D}(\tau) \cap AC_0(I)$, where $AC_0(I)$ are the functions in $AC(I)$ with compact support. Then we get the following result:

Theorem 1.1. *The closure of A_0 is given by*

$$\overline{A_0} u = \tau u, \quad \mathfrak{D}(\overline{A_0}) = \{u \in \mathfrak{D}(\tau) \ : \ W_0(u, v) = W_b(u, v) = 0, \ \forall v \in \mathfrak{D}(\tau)\} \,.$$

Its adjoint is given by

$$A_0^* u = \tau u, \quad \mathfrak{D}(A_0^*) = \mathfrak{D}(\tau).$$

Proof. [12, Thm.9.4.] □

To get τ self-adjoint, we need to work a little bit harder. Therefore, we use the terms limit circle and limit point:
We call τ *limit circle* at 0, if there is a $f \in \mathfrak{D}(\tau)$, such that $W_0(\overline{f}, f) = 0$ with $W_a(\overline{f}, u) = 0$ for at least one $u \in \mathfrak{D}(\tau)$, otherwise τ is called *limit point* at 0. Similarly for b.
We see, τ is limit point at 0 if and only if $W_0(u, v) = 0, \ \forall u, v \in \mathfrak{D}(\tau)$.

Finally, we have:

Theorem 1.2. *If τ is limit circle at 0, then we can find, as above, a $f \in \mathfrak{D}(\tau)$ with $W_0(\overline{f}, f) = 0$ and $W_0(\overline{f}, u) = 0$ for some $u \in \mathfrak{D}(\tau)$. Similarly, if τ is limit circle at b, let g be the analogous function there. Then, the operator*

$$\begin{array}{rccl} A & : & \mathfrak{D}(\tau) & \rightarrow & L^2(I, r\,dx) \\ & & u & \mapsto & \tau u \end{array} \tag{1.6}$$

with

$$\begin{aligned} \mathfrak{D}(A) = \{u \in \mathfrak{D}(\tau) \mid & \; W_0(f, u) = 0 \text{ if } \tau \text{ is limit circle at } 0 \\ & W_b(g, u) = 0 \text{ if } \tau \text{ is limit circle at } b\} \end{aligned} \tag{1.7}$$

is self-adjoint.

Proof. [12, Thm.9.6.] □

1.2 The spectral theorem for Sturm-Liouville-operators

We start with the following theorem:

Theorem 1.3. *Suppose $rv \in L^1_{loc}(I)$, then there exists a unique solution $u, pu' \in AC(I)$ of the differential equation*

$$(\tau - z)\, f = g, \qquad z \in \mathbb{C},$$

satisfying the initial condition $u(c) = \alpha, \ (pu')(c) = \beta$ for $\alpha, \beta \in \mathbb{C}$ and $c \in I$.

This clearly tells us, that also the solution space of the equation $\tau u = zu$, for $z \in \mathbb{C}$ is two dimensional, so we can find two linear independent solutions $u_1(z,x)$ and $u_2(z,x)$ of the differential equation, where u_1 satisfies the boundary conditions at $c \in \mathbb{R}^+$, and u_2 is chosen, such that $W(u_1,u_2) = 1$. They are therefore given by the following conditions

$$\begin{aligned} u_1(z,c) = 0,\ u_2(z,c) = 1 \text{ and} \\ p(c)u_1'(z,c) = 0,\ p(c)u_2'(z,c) = 1. \end{aligned} \tag{1.8}$$

All other solutions of $\tau u = zu$ are linear combinations of these two solutions. Be aware, that the fact, that u_1 satisfies the boundary conditions, does not mean coercive that u_1 is an eigenfunction, because u_1 must not lie in $L^2(I, r(x)\mathrm{d}x)$. Furthermore, we need the following result:

Theorem 1.4. *Suppose $z \in \rho(A)$, then there exists a solution $u_b(z,x)$, which is square integrable near b and which can be chosen holomorphic with respect to z, such that*

$$u_b(z,x)^* = u_b(z^*,x).$$

Proof. [12, Thm.9.7.] □

Our gaol is the calculation of the spectrum of our operator A. Therefore, we use a result, which is given by the spectral theorem, which implies, that there is an unitary operator U, which maps our self adjoint operator A to multiplication by λ:

$$U \ :\ L^2(I, r(x)\mathrm{d}x) \to L^2(\mathbb{R}, \mathrm{d}\mu), \quad (Uf)(\lambda) = \int_0^b u_1(\lambda, x)f(x)\mathrm{d}x. \tag{1.9}$$

So the object of desire is the measure μ, because it contains all the spectral information of the operator A. There is an essentially fact, which will help us to calculate it. Our solution space is two dimensional, see above, so it is spanned by our two solutions u_1 and u_2, now we can represent the solution u_b in the following form:

$$u_b(z,x) = u_2(z,x) + m_b(z)u_1(z,x). \tag{1.10}$$

This function m_b is known as the Weyl-Titchmarsh m-function. It is holomorphic in $\rho(A)$ and satisfies

$$m_b(z)^* = m_b(z^*).$$

We call a holomorphic function F a *Herglotz function* if it satisfies $F : \mathbb{C}_+ \to \mathbb{C}_+$. Now, we have the following results

Theorem 1.5. *The Weyl m-function is a Herglotz function and satisfies*

$$\mathrm{Im}(m_b(z)) = \mathrm{Im}(z) \int_0^b |u_b(z,x)|^2 r(x)\mathrm{d}x \tag{1.11}$$

$$m_b(z) = d + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) \mathrm{d}\mu(\lambda), \ \textit{where} \tag{1.12}$$

$$d = \mathrm{Re}(m_b(i)), \ \textit{and} \ \int_{\mathbb{R}} \left(\frac{1}{1+\lambda^2} \right) \mathrm{d}\mu(\lambda) = \mathrm{Im}(m_b(i)) \tag{1.13}$$

Moreover, μ is given via the Stieltjes inversion formula

$$\mu(\lambda) = \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{\delta}^{\lambda+\delta} \mathrm{Im}(m_b(\lambda + i\epsilon))\mathrm{d}\lambda. \tag{1.14}$$

Proof. [12, Thm.9.14] □

So, all the spectral information is hidden in the Weyl m-function.

We require above all the following result:

Theorem 1.6. *The set* $\{\lambda \in \mathbb{R} \,:\, \mathrm{Im}(m_b(\lambda)) > 0\}$ *is a support for* μ*, the set* $\{\lambda \in \mathbb{R} \,:\, \mathrm{Im}(m_b(\lambda)) = \infty\}$ *is a support for the singularly part and the set* $\{\lambda \in \mathbb{R} \,:\, 0 < \mathrm{Im}(m_b(\lambda)) < \infty\}$ *is a support for the absolutely continuous part of the measure* μ*.*

Proof. [12, Thm.3.16] □

Chapter 2

The connection of bounded Eigenfunctions and AC-Spectrum

This chapter presents a generalization of [9]. The result in [9], a connection between bounded eigenfunctions and the absolutely continuous spectrum is shown there for Schrödinger operators of the following form: $(Hu) := -(u)'' + V(x)u(x)$, $u(0) = 0$. We generalize this result here for a large class of Sturm-Liouville operators. Finally, we will show Weidmann's theorem for Schrödinger operators.

2.1 The main result

We start by considering Sturm-Liouville operators on a half line, which are given in (1.1),

$$(\tau u)(x) = \frac{1}{r(x)} \Big(- (pu')' (x) + V(x)u(x) \Big), \tag{2.1}$$

with the boundary condition

$$u(0) = 0, \tag{2.2}$$

on the space $L^2((0, \infty); r(x)\mathrm{d}x)$.
We require the following properties for our functions V, p and r:

- $\Gamma(\frac{V}{r}) \equiv \sup\limits_x \left(\int_{x-1}^{x+1} \left| \frac{V(y)}{r(y)} \right|^2 \mathrm{d}y \right) < \infty.$
- The function r has to be "nice" everywhere, which means in our case locally bounded, further it may not change to much, which means $\gamma := \gamma(r) = \sup\limits_x \sup\limits_{y \in [x+1, x-1]} \frac{r(y)}{r(x)} < \infty$. We also need, that $r(0) = 1$, which can be achieved easily by normalizing.

- Finally, it is not allowed, that $\frac{1}{p}$ goes faster to zero, in an averaged sense, than $r(x)$ goes to infinity, which means:
$\omega := \omega(p,r) = \inf_y \left[\left(\inf_{x\in[y-1,y+1]} r(x) \right) \int_{-1/2}^{1/2} \frac{1}{p(x)} \mathrm{d}x \right] > 0.$

For any $z \in \mathbb{C}$, we find, as I mentioned in the previous section, two solutions of the formal differential equation $Hu = Eu$ satisfying the boundary conditions:

$$u_1(z;0) = 0, \qquad p(0)u_1'(z;0) = 1$$

$$u_2(z;0) = 1, \qquad p(0)u_2'(z;0) = 0.$$

We want to show the following theorem:

Theorem 2.1. *On $S = \{E \in \mathbb{R} \mid u_1\sqrt{r}$ and $u_2\sqrt{r}$ are bounded on $[\,0,\infty\,)\}$ the spectral measure μ for H is purely absolutely continuous.*

This theorem means that $\mu(T) > 0$ for any $T \subset S$ with $|T| > 0$, where $|\cdot|$ is the Lebesgue measure.

Remark 2.2. *The corresponding result for Schrödinger operators H with $p = r = 1$ was first given in [9]. For more informations about similar results in the Schrödinger case, see also [3], [4], [5] and [8].*
The steps, we will take to proof Theorem 2.1, are analogue to that one in [9].

The proof for the required result uses the theory of Weyl m-functions. Theorem 1.4. tells us, that we can find a unique solution $u_+ \in L^2$ of the differential equation for each $z \in \mathbb{C}_+$, which we normalize according to

$$u_+(z;0) = 1. \tag{2.3}$$

We now use the representation given in (1.10),

$$u_+(z;x) = u_2(z;x) + m_+(E)u_1(z;x),$$

where m_+ is again the Weyl m-function. If we differentiate this equation, multiply it with $p(x)$ and set $x = 0$, we see

$$m_+(z) = p(0)u_+'(z;0). \tag{2.4}$$

By Theorem 1.6., we know, that μ_{ac} is supported on

$$\left\{ E \in \mathbb{R} \mid 0 < \lim_{\epsilon\downarrow 0} \mathrm{Im}(m_+(E+i\epsilon)) < \infty \right\}.$$

Hence, Theorem 2.1. is an immediate consequence of

Theorem 2.3. *Let $E \in S$. Then*

$$\liminf_{\epsilon\downarrow 0} \mathrm{Im} m_+(E+i\epsilon) \geq \frac{\text{const.}}{(2 + \frac{64\gamma^2}{\omega^2} + 8\gamma^2\Gamma(|V/r - E|))} \tag{2.5}$$

$$\limsup_{\epsilon\downarrow 0} |m_+(E+i\epsilon)| \leq p(0)^2 \frac{(2 + \frac{64\gamma^2}{\omega^2} + 8\gamma^2\Gamma(|V/r - E|))}{\text{const.}}. \tag{2.6}$$

To proof Theorem 2.3., we begin with:

Lemma 2.4. *If u obeys $-u'' + Vu = Eu$ then*

$$\frac{|(pu')(y)|^2}{r(y)} \leq \left[\frac{16}{\gamma^2\omega^2} + \frac{2}{\gamma^2}\Gamma(|\frac{V}{r} - E|)\right] \int_{x-1}^{x+1} |u(y)|^2 r(y)\mathrm{d}y. \qquad (2.7)$$

Proof. To verify this, we abbreviate

$$P(x) = \int_0^x \frac{dy}{p(y)}.$$

Then, we calculate

$$\int_0^x (pf')'(y)(P(x) - P(y))\mathrm{d}y = (pf')(y)(P(x) - P(y))|_0^x$$

$$+ \int_0^x (pf')(y)\frac{1}{p(y)}\mathrm{d}y = -(pf')(0)P(x) + f(x) - f(0)$$

and

$$\int_{-x}^0 (pf')'(y)(P(-x) - P(y))\mathrm{d}y = (pf')(y)(P(-x) - P(y))|_{-x}^0$$

$$+ \int_{-x}^0 (pf')(y)\frac{1}{p(y)}\mathrm{d}y = (pf')(0)P(-x) + f(0) - f(-x),$$

to get

$$(pf')(0) = \frac{f(x) - f(-x)}{P(x) - P(-x)} - \int_0^x (pf')'(y)\frac{P(x) - P(y)}{P(x) - P(-x)}\mathrm{d}y$$

$$+ \int_{-x}^0 (pf')'(y)\frac{-P(-x) + P(y)}{P(x) - P(-x)}\mathrm{d}y.$$

Now, use that $\frac{P(x)-P(y)}{P(x)-P(-x)} \leq 1$ for $0 \leq y \leq x$ and integrate from $\frac{1}{2}$ to 1. We obtain

$$|(pf')(0)| \leq 2\int_{\frac{1}{2}}^1 \frac{|f(x) - f(-x)|}{P(x) - P(-x)}\mathrm{d}x + 2\int_{\frac{1}{2}}^1 \int_{-x}^x |(pf')'(y)|\,\mathrm{d}y\mathrm{d}x$$

$$\leq 2\frac{1}{P(1/2) - P(-1/2)} \int_{-1}^1 |f(x)|\mathrm{d}x + \int_{-1}^1 |(pf')'(x)|\,\mathrm{d}x.$$

Multiply the last inequality on both sites with $\inf_{x\in[-1,1]} \frac{1}{\sqrt{r(x)}}$ and use, that there exists the constant γ, such that $\frac{1}{\sqrt{r(0)\gamma}} \leq \inf_{x\in[-1,1]} \frac{1}{\sqrt{r(x)}}$, to have

$$\frac{1}{r(0)}|(pf')(0)| \leq 2\gamma\frac{1}{P(1/2) - P(-1/2)} \int_{-1}^1 |f(x)|\frac{1}{\sqrt{r(x)}}\mathrm{d}x$$

$$+\gamma\int_{-1}^1 |(pf')'(x)|\,\frac{1}{\sqrt{r(x)}}\mathrm{d}x.$$

Now, set f(x) = u(x) and use, that $(pu')'(x) + (V(x) - Er(x)) = 0$, to get

$$\frac{1}{r(0)}|(pu')(0)| \leq \int_{-1}^1 \left[2\gamma\frac{1}{P(1/2) - P(-1/2)}\frac{1}{r(x)} + \gamma\left|\frac{V(x)}{r(x)} - E\right|\right] |u(x)|\sqrt{r(x)}\mathrm{d}x$$

$$\leq \int_{-1}^{1} \left[2\gamma \left(\frac{1}{P(1/2) - P(-1/2)} \sup_{x\in[-1,1]} \frac{1}{r(x)} \right) + \gamma \left| \frac{V(x)}{r(x)} - E \right| \right] |u(x)| \sqrt{r(x)} \mathrm{d}x$$

$$\leq \int_{-1}^{1} \left[\frac{2\gamma}{\omega} + \gamma \left| \frac{V(x)}{r(x)} - E \right| \right] |u(x)| \sqrt{r(x)} \mathrm{d}x$$

Next, set $u(x) = u(x+y)$, $p(x) = p(x+y)$, $r(x) = r(x+y)$ and $V(x) = V(x+y)$ and square both sites of the last inequality, this implies

$$|(pf')(y)|^2 r(y) \leq \int_{y-1}^{y+1} \left(\frac{8\gamma^2}{\omega^2} + 2\gamma^2 \left| \frac{V(x)}{r(x)} - E \right|^2 \right) \int_{y-1}^{y+1} |f(x)|^2 r(x) \mathrm{d}x$$

$$\leq \left[\frac{16\gamma^2}{\omega^2} + 2\gamma^2 \Gamma(|\frac{V}{r} - E|) \right] \int_{y-1}^{y+1} |f(x)|^2 r(x) \mathrm{d}x$$

□

Remark 2.5. *The proof of the Lemma in the Schrödinger case is easier in the sense, that you can omit all the estimates, which concerns* p *and* r*. Furthermore, the terms* γ *and* ω *are equal to one. We have to start there with Taylor's theorem for the functions* $f(x)$ *and* $f(-x)$*. The comparison of both equations gives again an estimate for* $f'(0)$ *and then the steps are analogue to that one given here. The final inequality in the Schrödinger case is* $|u'(x)|^2 \leq \left(16 + \frac{1}{2}\Gamma(|V - E|)\right) \int_{x-1}^{x+1} |u(y)|^2 \mathrm{d}y$*.*

This lemma tells us, if $E \in S$, $\frac{1}{\sqrt{r}} pu'$ is bounded as well. The general transfer matrix $T(E; x, 0)$ for solutions of $\tau u = u$ is defined by

$$T(E; x, 0) = \begin{pmatrix} p(x)u_1'(E; x) & p(x)u_2'(E; x) \\ u_1(E; x) & u_2(E; x) \end{pmatrix}$$

Next, set

$$T(E; x, y) = T(E; x, 0) T(E; y, 0)^{-1}.$$

Then, it is easy to check that

$$T(E; x, y) \begin{pmatrix} p(y)u'(y) \\ u(y) \end{pmatrix} = \begin{pmatrix} p(x)u'(x) \\ u(x) \end{pmatrix}. \tag{2.8}$$

But this transfer matrix must not be bounded in our case, because, neither u nor pu' have to be bounded. So we require a special transfer matrix, where we know that it will be bounded then. Set

$$\widetilde{T}(E; x, 0) = \begin{pmatrix} \frac{1}{\sqrt{r(x)}} & 0 \\ 0 & \sqrt{r(x)} \end{pmatrix} T(E; x, 0). \tag{2.9}$$

Then set as before

$$\widetilde{T}(E; x, y) = \widetilde{T}(E; x, 0) \widetilde{T}(E; y, 0)^{-1},$$

to get

$$\widetilde{T}(E;x,y)\begin{pmatrix}\frac{1}{\sqrt{r(y)}}p(y)u'(y)\\ \sqrt{r(y)}u(y)\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{r(x)}}p(x)u'(x)\\ \sqrt{r(x)}u(x)\end{pmatrix}. \tag{2.10}$$

For $E\in S$, this transfer matrix is bounded, so we can define the following value

$$C(E)\equiv\sup_{x,y}\left\|\widetilde{T}(E;x,y)\right\|, \tag{2.11}$$

which is finite.

Proof of theorem 2.2. Suppose the function ϕ satisfies $-(p\phi')'/r+(V/r-E-i\epsilon)\phi=0$. Next, we rewrite this equation as $-(p\phi')'/r+(V/r-E)\phi=i\epsilon\phi$ and use variation of constants.
Let $\underline{\phi}=\begin{pmatrix}p\phi'\\ \phi\end{pmatrix}$, then

$$\underline{\phi}(x)=T(E;x,0)\underline{\phi}(0)+i\epsilon\int_0^x T(E;x,y)\underline{\phi}(y)\mathrm{d}y.$$

Multiply both sites with $\begin{pmatrix}\frac{1}{\sqrt{r(x)}} & 0\\ 0 & \sqrt{r(x)}\end{pmatrix}$ and set $\widetilde{\underline{\phi}}=\begin{pmatrix}\frac{1}{\sqrt{r}}p\phi'\\ \sqrt{r}\phi\end{pmatrix}$, to have

$$\widetilde{\underline{\phi}}(x)=\widetilde{T}(E;x,0)\widetilde{\underline{\phi}}(0)+i\epsilon\int_0^x \widetilde{T}(E;x,y)\widetilde{\underline{\phi}}(y)\mathrm{d}y.$$

Observe, that $\widetilde{\underline{\phi}}(0)=\underline{\phi}(0)$, because $r(0)=1$. The last equation implies

$$\left\|\widetilde{\underline{\phi}}(x)\right\|\le C(E)\left\|\widetilde{\underline{\phi}}(0)\right\|+C(E)\epsilon\int_0^x\left\|\widetilde{\underline{\phi}}(y)\right\|\mathrm{d}y.$$

Because of the Gronwall Lemma, we get

$$\left\|\widetilde{\underline{\phi}}(x)\right\|\le C(E)e^{\epsilon C(E)|x|}\left\|\widetilde{\underline{\phi}}(0)\right\|$$

and therefore

$$\left\|\widetilde{T}(E+i\epsilon,x,0)\right\|\le C(E)e^{\epsilon C(E)|x|}. \tag{2.12}$$

Next, we use our square integrable solution $u_+(E+i\epsilon,x)$ and Lemma 2.4. to obtain

$$\frac{1}{r(x)}|u_+'(x)|^2\le\left[\frac{16\gamma^2}{\omega^2}+2\gamma^2\Gamma(|\frac{V}{r}-E-i\epsilon|)\right]\int_{x-1}^{x+1}|u_+(y)|^2r(y)\mathrm{d}y.$$

Now, integrate this from 1 to ∞ to have

$$\int_1^\infty\frac{1}{r(x)}|u_+'(x)|^2\mathrm{d}x\le\left[\frac{32\gamma^2}{\omega^2}+4\gamma^2\Gamma(|V-E-i\epsilon|)\right]\int_0^\infty|u_+(y)|^2r(y)\mathrm{d}y.$$

Next set $\beta=1/\left(1+\frac{32\gamma^2}{\omega^2}+4\gamma^2\Gamma(|V-E-i\epsilon|)\right)$ to have

$$\int_0^\infty r(y)|u_+(y)|^2\mathrm{d}y\ge\beta\int_1^\infty\left[r(x)|u_+(x)|^2+\frac{1}{r(x)}|u_+'(x)|^2\right]\mathrm{d}x$$

$$= \beta \int_1^\infty \left\| \widetilde{T}(E+i\epsilon;x,0)\underline{\widetilde{u_+}}(0)\right\|^2 \mathrm{d}x$$

$$\geq \frac{\beta}{C(E)^2}\left\|\underline{\widetilde{u_+}}(0)\right\|^2 \int_1^\infty e^{-2\epsilon C(E)x}\mathrm{d}x = \frac{\beta e^{-2\epsilon C(E)}}{2C(E)^3\epsilon}\left(1+|\frac{1}{p(0)}m_+(E+i\epsilon)|^2\right).$$

We obtain the second inequality in the last equation from (2.11) and from the constancy of the Wronskian. To see this, suppose there is a solution $\overline{v}$, which decrease faster than any other solution $\overline{u}$ can increase. We know, that the Wronskian of two solutions is constant. If we value the Wronskian at ∞, we see, that $W(\overline{v},\overline{u}) = 0$, but we can choose the function $\overline{u}$ arbitrarily and in particular linearly independent of $\overline{v}$, which implies $\overline{u} = 0$, a contradiction. Multiply both sides with $\mathrm{Im}(E + i\epsilon)$ to get

$$\mathrm{Im}(m_+(E+i\epsilon)) = \epsilon \int_0^\infty |u_+(y)|^2 r(y)\mathrm{d}y \geq \frac{\beta e^{-2\epsilon C(E)}}{2C(E)^3}\left(1+|\frac{1}{p(0)}m_+(E+i\epsilon)|^2\right).$$

And therefore

$$\liminf_{\epsilon\downarrow 0}\left[\frac{\mathrm{Im}(m_+(E+i\epsilon))}{\left(1+|\frac{1}{p(0)}m_+(E+i\epsilon)|^2\right)}\right] \geq \frac{\beta}{2C(E)^3}.$$

Because of $\left(1+|\frac{1}{p(0)}m_+(E+i\epsilon)|^2\right) \geq 1$, we immediately see, that

$$\liminf_{\epsilon\downarrow 0}\mathrm{Im}(m_+(E+i\epsilon)) \geq \frac{\beta}{2C(E)^3}, \tag{2.13}$$

which is (2.5) Now, to obtain (2.6) use $\frac{\left(1+|\frac{1}{p(0)}m_+|^2\right)}{\mathrm{Im}(m_+)} \geq \frac{1}{p(0)^2}|m_+|$, which can easily be verified, to have

$$\left(\frac{\beta}{2C(E)^3}\right)^{-1} \geq \limsup_{\epsilon\downarrow 0}\left[\frac{\left(1+|\frac{1}{p(0)}m_+(E+i\epsilon)|^2\right)}{\mathrm{Im}(m_+(E+i\epsilon))}\right] \geq \limsup_{\epsilon\downarrow 0}\frac{1}{p(0)^2}|m_+(E+i\epsilon)|. \tag{2.14}$$

□

Remark 2.6. *The inequalities for the Schrödinger operator H are given by* $\liminf_{\epsilon\downarrow 0}\mathrm{Im}(m_+(E+i\epsilon)) \geq \frac{1}{2}\mathrm{const.}/\left(33+\Gamma(|V-E|)\right)$ *and* $\limsup_{\epsilon\downarrow 0}|m_+(E+i\epsilon)| \leq \left[\frac{1}{2}\mathrm{const.}/\left(33+\Gamma(|V-E|)\right)\right]^{-1}$.

2.2 Weidmann's Theorem

In this section, we want to establish Weidmann's Theorem by showing how we can find the asymptotics of the eigenfunctions in the Schrödinger case, which means for the operator H. The facts here are from [9].

Theorem 2.7. *Let $V(x) = V_1(x) + V_2(x)$, where $V_1 \in L^1$ and $V_2 \in C^1$ with $V_2' \in L^1$ and $V_2(x) \overset{x\to\infty}{\to} 0$. Next, fix $E = k^2 > 0$ with $k > 0$. Then every solution of*

$$\left(-\frac{d^2}{dx^2} + V(x)\right) u = Eu \tag{2.15}$$

is bounded. Indeed, there exists a and b so that

$$|u(x) - au_+(x) - bu_-(x)| \to 0 \tag{2.16}$$

$$|u'(x) - iaku_+(x) + ibku_-(x)| \to 0, \tag{2.17}$$

where

$$u_\pm(x) = \exp\left(\pm i \int_{x_0}^{x} \sqrt{k^2 - V_2(x)}\, dx\right) \tag{2.18}$$

and x_0 is chosen so large that $V_2(x) \le k^2$ for $x \ge x_0$.

Proof. First of all notice, that

$$\begin{aligned}
- &\quad u''_\pm(x) + (V(x) - E)u_\pm(x) = \\
= &\quad -\left[u_\pm(x)\left(\pm i\sqrt{k^2 - V_2(x)}\right)^2 + u_\pm(x)\left(\mp i \frac{1}{2}\frac{V_2'(x)}{\sqrt{k^2 - V_2(x)}}\right)\right] \\
+ &\quad (V(x) - E)u_\pm(x) \\
= &\quad u_\pm(x)\left[\left(k^2 - V_2(x)\right) \pm \frac{i}{2}\frac{V_2'(x)}{\sqrt{k^2 - V_2(x)}} + V(x) - E\right] \\
= &\quad u_\pm(x)\left[V_1(x) \pm \frac{i}{2}V_2'(x)\left(\sqrt{k^2 - V_2(x)}\right)^{-1}\right].
\end{aligned}$$

We now define

$$F_\pm(x) := V_1(x) \pm \frac{i}{2}V_2'(x)\left(\sqrt{k^2 - V_2(x)}\right)^{-1}. \tag{2.19}$$

From the conditions, it follows, that this term is L^1.

For the Wronskian $W_x(u_+, u_-)$ of u_+ and u_-, we have:

$$W_x = W_x(u_+, u_-) = 2i\sqrt{k^2 - V_2(x)}.$$

So, because of the continuity of V_2 and the condition $V_2 \to 0$, it follows:

$$W_x = 2ki + o(1). \tag{2.20}$$

Next, define $a(x)$ and $b(x)$ through the equations

$$\begin{aligned}
u(x) &= a(x)u_+(x) + b(x)u_-(x) \\
u'(x) &= a(x)u_+'(x) + b(x)u_-'(x).
\end{aligned} \tag{2.21}$$

If we differentiate the first one, and compare this with the second, we see

$$a'u_+(x) + a(x)u_+'(x) + b'(x)u_-(x) + b(x)u_-'(x) = a(x)u_+(x) + b(x)u_-(x)$$

and therefore

$$a'(x) = -b'(x)u_-^2(x) \text{ rsp. } b'(x) = -a'u_+^2(x).$$

Because $\left(-\frac{d^2}{dx^2} + V(x)\right) u = Eu$, it also follows

$$\begin{aligned}
0 = &- \left(a'(x)u_+'(x) + b'(x)u_-'(x) + a(x)u_+''(x) + b(x)u_-''(x)\right) \\
&+ (V(x) - E)\,(a(x)u_+(x) + b(x)u_-(x)) \\
&= b'(x)\left[u_-^2(x)u_+'(x) - u_-'(x)\right] \\
&+ a(x)\left(-u_+'' + V(x)u_+(x) - Eu_+(x)\right) + b(x)\left(-u_-'' + V(x)u_-(x) - Eu_-(x)\right) \\
&= b'(x)u_-(x)\left[2i\sqrt{k^2 - V_2(x)}\right] + a(x)F_+(x)u_+(x) + b(x)F_-(x)u_-(x)
\end{aligned}$$

and altogether

$$b'(x) = \frac{1}{W_x}\left(-F_+(x)u_+^2(x)a(x) - F_-(x)b(x)\right)$$

respectively

$$a'(x) = \frac{1}{W_x}\left(F_+(x)a(x) + F_-(x)u_-^2(x)b(x)\right).$$

We will write this in matrix form

$$\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}' = M(x)\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}, \tag{2.22}$$

where

$$M(x) := W_x^{-1}\begin{pmatrix} F_+(x) & u_-(x)^2F_-(x) \\ -F_+(x)u_+^2(x) & -F_-(x) \end{pmatrix}. \tag{2.23}$$

The Wronskian W_x is bounded, $F_\pm \in L^1$ and $|u_\pm(x)| = 1$, so $\|M(x)\| \in L^1$ follows. It further follows, that:

$$\left\|\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}'\right\| \leq \|M(x)\|\left\|\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}\right\|. \tag{2.24}$$

Now, let $c(x) = \left\|\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}\right\|$, then it follows by Gronwall, that

$$c(x) \leq c(x_0)\exp\left(\int_{x_0}^{x} \|M(s)\| \text{ ds}\right) \leq c(x_0)\exp\left(\int_{x_0}^{\infty} \|M(s)\| \text{ ds}\right) < \infty,$$

for $x > x_0$.

Next, we want to show that $\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$ is a Cauchy sequence, then it would follow that the limit $\lim_{x\to\infty}\begin{pmatrix} a(x) \\ b(x) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ exists. To see that $\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$ is indeed a Cauchy sequence consider the following inequality:

$$\left\|\begin{pmatrix} a(x_2) \\ b(x_2) \end{pmatrix} - \begin{pmatrix} a(x_1) \\ b(x_1) \end{pmatrix}\right\| \leq c(x_0)\exp\left(\int_{x_0}^{\infty} \|M(s)\|\,\text{ds}\right)\exp\left(\int_{x_1}^{x_2} \|M(s)\|\,\text{ds}\right).$$

□

Chapter 3

The Laplacian on a regular tree

The following chapter gives the full description of regular trees, of the Laplacian on it and finally, of the spectral properties of the Laplacian on a regular tree. The main facts are taken from [10] and [7].

3.1 The regular rooted tree

I now want to establish the idea of a rooted metric tree Γ. Our tree consists of the root o and infinitely many edges $E = E(\Gamma)$ and Vertices $V = V(\Gamma)$. For any two points $x, y \in \Gamma$, there exists an unique shortest path $\langle x, y \rangle$ from x to y. We write $x \prec y$, if $x \in \langle o, y \rangle$ and $x \neq y$. Next, for a vertex $\nu \in V$, we define its generation $\mathrm{gen}(\nu)$, by

$$\mathrm{gen}(\nu) = \sharp \{\mu \in V(\Gamma) : \mu \prec \nu\}.$$

This in particular means, $\nu = o$ is the only vertex such that $\mathrm{gen}(\nu) = 0$.
For any edge $\epsilon = \langle \nu, \mu \rangle$, for $\nu, \mu \in V$ and $\nu \prec \mu$, we set $\mathrm{gen}(\epsilon) = \mathrm{gen}(\nu)$. In our tree, all edges of the same generation have the same length. This means, that the distance to the root is equal for all vertices of the same generation. Therefore, we can describe the distance of the vertices from the root of the tree in an easy sequence:

$$|\nu| = t_{gen(\nu)} \qquad \forall \nu \in V(\Gamma), \tag{3.1}$$

where $|\nu|$ means the distance of the vertex ν to the root.
Finally, we have to define the branching number $b(\nu)$ on each vertex $\nu \in V$. Here, we look at trees, where the branching number has to be equal for vertices of the same generation. This clearly means:

$$b(\nu) = b_{gen(\nu)} \qquad \forall \nu \in V(\Gamma). \tag{3.2}$$

Altogether, our trees are fully determined by the two sequences:

$$\{b_n\} = \{b_n(\Gamma)\} \text{ and } \{t_n\} = \{t_n(\Gamma)\}.$$

We consider only trees, where $b_0 = 1$ and $b_k \geq 2$ for $k \geq 1$. So, a tree is said to be *regular*, if it satisfies all these properties.

From the definition of the value t_k, it is clear, that $t_0 = 0$ and the sequence $\{t_k\}$ is strictly increasing. Now define

$$h_\Gamma = \lim_{n\to\infty} t_n. \tag{3.3}$$

It is natural to refer to h_Γ as the high of the tree. Next, we establish the so called branching function $g_\Gamma(t)$

$$g_\Gamma(t) = \sharp\{x \in \Gamma \ : \ |x| = t\}. \tag{3.4}$$

It is clear, that g_Γ is of the form

$$g_\Gamma(t) = \begin{cases} b_1 \dots b_n, & t_n < t \leq t_{n+1}, \\ 1, & 0 \leq t \leq t_1. \end{cases}$$

We want to define the branching function for subtrees $T \subset \Gamma$ too, but there are two types of subtrees, which are associated with vertices on the one hand and with edges on the other hand, so let $\epsilon = \langle \nu, \mu \rangle \in E(\Gamma)$, with $\nu, \mu \in V(\Gamma)$ and $\nu \prec \mu$, set

$$T_\mu = \{x \in \Gamma \ : \ x \succ \mu\} \cup \{\mu\}, \quad T_\epsilon = \epsilon \cup T_\mu. \tag{3.5}$$

Up to this point, it is clear, that $T_o = T_{\epsilon_o} = \Gamma$. Now let $T = T_\epsilon$ and $\text{gen}(\epsilon) = \text{gen}(\nu) = k$, then we will get $g_T(t) = g_k(t)$, where

$$g_k(t) = \begin{cases} 0, & 0 \leq t < t_k, \\ 1, & t_k \leq t \leq t_{k+1}, \\ b_{k+1} \dots b_n, & t_n < t \leq t_{n+1}. \end{cases} \tag{3.6}$$

Therefore, we get the simple connections: $g_k(t) = (b_0 \cdots b_k)^{-1} g_\Gamma(t)$ for $t \geq t_k$ and $g_{T_\nu} = b_k g_k(t)$, if $\text{gen}(\nu) = k$.

3.2 Decomposition of the space $M(\Gamma)$

We start with the definition of the required spaces: We denote by $M(\Gamma)$ the linear space of all measurable functions that are finite almost everywhere. Let $M_c(\Gamma)$ be the space of all functions in $M(\Gamma)$ supported by only finely many edges.

Two functions $u, v \in M(\Gamma)$ are said to be level wise orthogonal, if they satisfy

$$\sum_{x\in\Gamma:|x|=t} u(x)\overline{v(x)} = 0 \text{ for almost all } t > 0. \tag{3.7}$$

Two subspaces $F, G \subset M(\Gamma)$ are level wise orthogonal if the above relation holds for any $u \in F$ and $v \in G$. For a subtree $T \subset \Gamma$, a function $u \in M(\Gamma)$ belongs to the class $\mathfrak{M}_T$ if

$$u(x) = 0, \text{ if } x \notin T, \quad u(x) = u(y) \text{ if } x, y \in T \text{ and } |x| = |y|. \tag{3.8}$$

This clearly means, we can identify any such function $u \in \mathfrak{M}_T$ with the corresponding function $f = J_T u \in M(|o_T|, h_\Gamma)$, where J_T is a natural embedding of

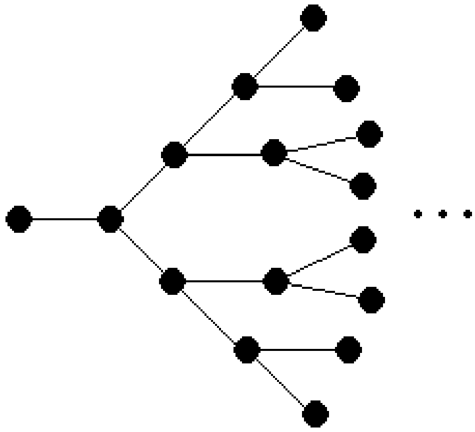

Figure 3.1: Regular tree graph with constant edge length and branching number $b_n = 2$, for $n \geq 1$.

the classes $\mathfrak{M}_T$ to the spaces $M(|o_T|, h_\Gamma)$, such that $u(x) = f(|x|)$ for $x \in T$ almost everywhere. $M(I)$ stands for the space of measurable functions, that are finite almost everywhere. This implies, that the subspace $\mathfrak{M}_\Gamma$ consists of all symmetric functions on Γ. We now want to find an operator, which acts as a projection from the space $M(\Gamma)$ onto $\mathfrak{M}_T$. This operator is given by:

$$(P_T u)(x) = \begin{cases} g_T(|x|)^{-1} \sum_{y \in T:|y|=|x|} u(y) & \text{for } x \in T, \\ 0 & \text{for } x \notin T \end{cases} . \tag{3.9}$$

We can say, that P_T defines a level wise orthogonal projection onto $\mathfrak{M}_T$.

We, also need the subspaces $\mathfrak{M}_T$ associated with our subtrees T_ν and T_ϵ. We start with $\text{gen}(\nu) = k$, for simplicity we write $\mathfrak{M}_\nu$ instead of $\mathfrak{M}_{T_\nu}$ and $\mathfrak{M}_\nu^j$ instead of $\mathfrak{M}_{T_{\epsilon_\nu^j}}$, $j = 1, \dots, b_k$. $T_{\epsilon_\nu^j}$ stands for the j-th edge starting at ν. So for a given $u \in M(\Gamma)$, we define the functions $f_{\nu,u}, f_{\nu,u}^j \in M(t_k, h_\Gamma)$ as follows:

$$f_{\nu,u} := J_{T_\nu} P_{T_\nu} u,$$

$$f_{\nu,u}^j := J_{T_{\epsilon_\nu^j}} P_{T_{\epsilon_\nu^j}} u, \text{ for } j = 1, \dots, b_k.$$

It can be immediately proved, that the spaces $\mathfrak{M}_\nu^j$ for $j = 1, \dots, b_k$ are level wise orthogonal, so we take a closer look at their linear sum $\widetilde{\mathfrak{M}_\nu} = \sum_{j=1}^{b_k} \mathfrak{M}_\nu^j$, which contains the whole space $\mathfrak{M}_\nu$.

Any function $u = \{u_1, \dots, u_{b_k}\} \in \widetilde{\mathfrak{M}_\nu}$ can be identified with a vector-valued function $\mathbf{f} = \widetilde{J_\nu} u \in (M(t_k, h_\Gamma))^{b_k}$ realized by

$$\mathbf{f} = \{f_1, \dots, f_{b_k}\}, \text{ where } f_j = J_{T_{\epsilon_\nu^j}} u_j, \quad j = 1, \dots, b_k. \tag{3.10}$$

For the further analysis, we need to extend $\mathbf{f}$ with an other basis. We will use the discrete Fourier transform. So, we write $\omega = e^{(2\pi i)/b_k}$ and set

$$\mathbf{h}^{\langle s\rangle} = \frac{1}{\sqrt{b_k}}\left\{\omega^s, \dots, \omega^{s(b_k-1)}, 1\right\}, \quad s = 1, \dots, b_k. \tag{3.11}$$

It is straightforward to check, that these vectors form an orthogonal basis in $\mathbb{C}^{b_k}$. Next, to any s and any function $f \in M(t_k, h_\Gamma)$, we assign the vector valued function

$$\mathbf{f}^{\langle s\rangle} = \mathbf{h}^{\langle s\rangle} f \in (M(t_k, h_\Gamma))^{b_k}.$$

By the Fourier Transform, it follows that

$$\left\|\mathbf{f}^{\langle s\rangle}(t)\right\| = |f(t)|, \text{ almost everywhere on } (t_k, h_\Gamma). \tag{3.12}$$

Now, choose a s, then the set of functions $\widetilde{J_\nu}^{-1}\mathbf{f}^{\langle s\rangle}$ is a proper subspace of $\widetilde{\mathfrak{M}_\nu}$. Set

$$\widetilde{J_\nu}^{-1}\mathbf{f}^{\langle s\rangle} = \mathfrak{M}_\nu^{\langle s\rangle}.$$

By the construction, the spaces $\mathfrak{M}_\nu^{\langle s\rangle}$ do not only span the space $\widetilde{\mathfrak{M}_\nu}$, but also, they are level wise orthogonal.

The level wise orthogonal projections of $M(\Gamma)$ onto the subspaces $\mathfrak{M}_\nu^{\langle s\rangle}$ are given by

$$\widetilde{J_\nu} P_\nu^{\langle s\rangle} u = \mathbf{h}^{\langle s\rangle} f_\nu^{\langle s\rangle}, \text{ where } f_\nu^{\langle s\rangle} = \frac{1}{\sqrt{b_k}} \sum_{j=1}^{b_k} f_{\nu,u}^j \omega^{-js}. \tag{3.13}$$

If we set in the last equation $s = b_k$, it follows from the construction, that $\mathbf{h}^{\langle b_k\rangle} = \frac{1}{\sqrt{b_k}}\{1, \dots, 1\}$, so it is obvious, with the help of equation (3.13) that $\mathfrak{M}_\nu^{\langle b_k\rangle}$ coincides with $\mathfrak{M}_\nu$.

Next, we will introduce the following space

$$\mathfrak{M}'_\nu := \left\{ u \in \widetilde{\mathfrak{M}_\nu} \; : \sum_{x \in T_\nu : |x| = t} u(x) = 0 \text{ for almost all } t > 0 \right\}. \tag{3.14}$$

Clearly the subspaces $\mathfrak{M}_\nu$ and $\mathfrak{M}'_\nu$ are level wise orthogonal. Because $\mathfrak{M}_\nu$ and $\mathfrak{M}_\nu^{\langle s\rangle}$, for $s = 1, \dots, (b_k - 1)$, are level wise orthogonal and all together span the whole space $\widetilde{\mathfrak{M}_\nu}$, it follows, that

$$\mathfrak{M}'_\nu = \sum_{j=1}^{b_k-1} \mathfrak{M}_\nu^{\langle s\rangle}.$$

Now, we are able to summarize all the information, we have collected up to this point in a theorem

Theorem 3.1. *We have the following statements*

- *The subspaces* $\mathfrak{M}_\nu^{\langle s\rangle}$, $\nu \in V \setminus \{o\}$, $s = 1, \dots, b(\nu) - 1$ *and the subspace* $\mathfrak{M}_\Gamma$ *are mutually level wise orthogonal.*

- *One has*

$$M_c(\Gamma) \subset \mathfrak{M}_\Gamma \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{\mathrm{gen}(\nu)=k} \bigoplus_{s=1}^{b_k-1} \mathfrak{M}_z^{\langle s\rangle} \tag{3.15}$$

and for an arbitrary $u \in M_c(\Gamma)$, one has

$$\sum_{x\in\Gamma:|x|=t} |u(x)|^2 = |f_\Gamma(t)|^2 + \sum_{k=1}^{\infty} \sum_{\mathrm{gen}(\nu)=k} \sum_{s=1}^{b_k-1} |f_\nu^{\langle s\rangle}(t)|^2 \tag{3.16}$$

almost everywhere on $(0, h_\Gamma)$, where $f_\Gamma(t) = f_{o,u}(t)$ and the functions $f_\nu^{\langle s\rangle}(t)$ are given in (3.13). Let me remark that the sum on the right site of (3.16) is finite, since there is an k, such that all $\nu \in V$ with gen$(\nu) \geq k$, such that $f_\nu^{\langle s\rangle}(t) = 0$, for $t \geq k$, because u is supported on only finitely many edges.

Proof. Instead of proofing, that the subspaces $\mathfrak{M}_\Gamma$ and $\mathfrak{M}_\nu^{\langle s\rangle}$, $\nu \in V \setminus \{o\}$, $s = 1, \ldots, b(\nu) - 1$ are level wise orthogonal, we show, that $\mathfrak{M}_\Gamma$ and $\mathfrak{M}'_\nu$, for $\nu \in V \setminus \{o\}$ are level wise orthogonal, but this follows from (3.8), for $T = \Gamma$ and (3.14). Next let $\mu, \nu \in V \setminus \{o\}$ and let neither $\mu \prec \nu$ nor $\nu \prec \mu$, but then, the corresponding subtrees T_ν and T_μ are disjoint, which implies level wise orthogonality for the subspaces $\mathfrak{M}'_\mu$ and $\mathfrak{M}'_\nu$. Further, let $\nu \prec \mu$ and $u \in \mathfrak{M}'_\nu$ and $v \in \mathfrak{M}'_\mu$, then, we know that u is symmetric on each subtree $T_{\epsilon_\nu^j}$, for $j = 1, \ldots, b(\nu)$, while, the function v vanishes outside the subtree T_μ, but this subtree is completely contained in some subtree $T_{\epsilon_\nu^j}$. Therefore, we obtain

$$\sum_{x\in T_\nu:|x|=t} u(x)\overline{v(x)} = \sum_{x\in T_\mu:|x|=t} u(x)\overline{v(x)} = \mathrm{const}(t) \sum_{x\in T_\mu:|x|=t} \overline{v(x)} = 0$$

Finally, the fact, that the subspaces $\mathfrak{M}_\mu^{\langle s\rangle}$ for different values s are level wise orthogonal is already known.

The proof of the second part of the theorem can be seen in [7, Thm.2.2.] ☐

Now, we take the Hilbert space $L^2(\Gamma)$, where

$$\|u(x)\|^2_{L^2(\Gamma)} = \int_\Gamma |u(x)|^2 \,\mathrm{d}x := \sum_{\epsilon\in E} \int_{t_{gen(\epsilon)}}^{t_{gen(\epsilon)+1}} |u(t)|^2 \,\mathrm{d}t \tag{3.17}$$

is finite, for every $u \in L^2(\Gamma)$. Set

$$L_\Gamma = \mathfrak{M}_\Gamma \cap L^2(\Gamma) \text{ and } L_\nu^{\langle s\rangle} = \mathfrak{M}_\nu^{\langle s\rangle} \cap L^2(\Gamma) \quad \text{for } \nu \in V \setminus \{o\} \text{ and } s = 1, \ldots, b(\nu) - 1. \tag{3.18}$$

This spaces are mutually orthogonal subspaces of $L^2(\Gamma)$.

Theorem 3.2. *The subspaces, given in (3.18) are mutually orthogonal in the space $L^2(\Gamma)$ and define an orthogonal decomposition of this space. For any function $u \in L^2(\Gamma)$, on has*

$$\int_\Gamma |u(x)|^2 \mathrm{d}x = \int_0^{h_\Gamma} |f_\Gamma(t)|^2 g_\Gamma(t)\,dt + \sum_{k=1}^{\infty} \sum_{\mathrm{gen}(\nu)=k} \sum_{s=1}^{b_k-1} \int_{t_k}^{h_\Gamma} |f_\nu^{\langle s\rangle}(t)|^2 g_k(t) \mathrm{d}t. \tag{3.19}$$

The functions $f_\Gamma(t)$ and $f_\nu^{\langle s\rangle}(t)$ are given as above. In particular,

$$L^2(\Gamma) = L_\Gamma \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{\text{gen}(\nu)=k} \bigoplus_{s=1}^{b_k-1} L_z^{\langle s\rangle}. \tag{3.20}$$

Proof. The Theorem is an immediate consequence of Theorem 3.1. □

3.3 Reduction of the Laplacian

We start with the definition of two required spaces

- $u \in H = H(\Gamma)$, if $u(o) = 0$, $u|_\epsilon \in H^1(\epsilon)$, $\forall \epsilon \in E$ and

$$\|u\|_H^2 := \int_\Gamma |u'|^2 \mathrm{d}x < \infty \tag{3.21}$$

  and

- $u \in H^{1,o}(\Gamma)$, if $u(0) = 0$

$$\|u\|_{H^1}^2 := \int_\Gamma \left(|u'(x)|^2 + |u(x)|^2\right) \mathrm{d}x. \tag{3.22}$$

The space H can be introduced, by the function, which allow the following representation:

$$u(x) = \int_{\langle 0,x\rangle} v(y)\mathrm{d}y, \text{ for } v \in L^2(\Gamma) \tag{3.23}$$

In this representation, it is clear, that one has $v = u'$ and that the mapping $u \to u'$ defines a natural isometry of H onto $L^2(\Gamma)$. But, let me remark, that a function $u \in H$, need not belong to $L^2(\Gamma)$

Next, we define the following closed subspaces of H

$$H_\Gamma = \mathfrak{M} \cap H, \quad H_\nu^{\langle s\rangle} = \mathfrak{M}_\nu^{\langle s\rangle} \cap H, \quad \nu \in V \setminus \{o\}, \quad s = 1, \ldots, b(\nu) - 1. \tag{3.24}$$

The operator P_Γ acts on H and defines there an orthogonal projection onto H_Γ. The similar assertion holds for the operators $P_\nu^{\langle s\rangle}$ for $s = 1, \ldots, b(\nu) - 1$, indeed for any function $u \in H$, the function $v = P_\nu^{\langle s\rangle} u$ has a derivative v' and $f_\nu^{\langle s\rangle}(t_k+) = 0$, and therefore it is continuous on all of Γ. Therefore, this ensures the relation $v = P_\nu^{\langle s\rangle} u \in H$. Furthermore differentiation preserves the classes $\mathfrak{M}_\nu^{\langle s\rangle} u$. This finally leads to the fact, that the operator $P_\nu^{\langle s\rangle} u$ defines an orthogonal projection from H onto $H_\nu^{\langle s\rangle}$. Let me mention, that the same facts for the operators P_T do not hold, because of the discontinuity at 0_T.

We can now formulate the following result:

Theorem 3.3. *Let $h_\Gamma = \infty$, then the subtrees defined in (3.24) are mutually orthogonal in H and define its orthogonal decomposition. For any function $u \in H$, we have the following equation*

$$\int_\Gamma |u'(x)|^2 \mathrm{d}x = \int_0^{h_\Gamma} |\frac{\mathrm{d}f_\Gamma(t)}{\mathrm{d}t}|^2 g_\Gamma(t)dt + \sum_{k=1}^{\infty} \sum_{\text{gen}(\nu)=k} \sum_{s=1}^{b_k-1} \int_{t_k}^{h_\Gamma} |\frac{\mathrm{d}f_z^{\langle s\rangle}(t)}{\mathrm{d}t}|^2 g_k(t)dt. \tag{3.25}$$

A similar statement holds for the space $H^{1,o}(\Gamma)$.

Proof. The result follows from Theorem 3.2. □

The next goal is to understand the nature of the operator $\Delta|_{\mathfrak{M}_\Gamma}$, respectively of each operator $\Delta|_{\mathfrak{M}_\nu^{\langle s\rangle}}$ for $s = 1, \dots, (b(\nu) - 1)$.

We start with the weighted Hilbert space $\mathfrak{H}_k := L^2((t_k, \infty); g_k)$ and the following quadratic form in it:

$$\alpha_k[\phi] = \int_{t_k}^{\infty} |\phi'(t)|^2 g_k(t)\mathrm{d}t, \qquad \phi \in H^{1,o}((t_k, \infty); g_k). \tag{3.26}$$

The quadratic form is non-negative and closed in $\mathfrak{H}_k$. Let U_k be the corresponding self adjoint operator, then, we have

Theorem 3.4. *The part of the operator Δ in the reducing space $\mathfrak{M}_\Gamma$ is unitarily equivalent to the operator U_0. Let $gen(\nu) > 0$ and $1 \le s < b(\nu)$, then, the part of the operator Δ in the reducing space $\mathfrak{M}_\nu^{\langle s\rangle}$ is unitarily equivalent to the operator U_k.*

Proof. [10, Thm.3.3.] Set $\widetilde{J_\nu^{\langle s\rangle}} := \widetilde{J_\nu}|_{\mathfrak{M}_\nu^{\langle s\rangle}}$. The result follows from the fact, that the operator $\widetilde{J_\nu^{\langle s\rangle}}$ maps the set $\mathfrak{M}_\nu^{\langle s\rangle} \cap H^{1,o}(\Gamma)$ onto $H^{1,o}((t_k, h_\Gamma); g_k)$, where $k = \mathrm{gen}(\nu)$ and because $f \in \mathfrak{M}_\nu^{\langle s\rangle} \cap H^{1,o}(\Gamma)$ implies $\int_\Gamma |f'(x)|^2 \mathrm{d}x = \alpha_k[J_\nu^{\langle s\rangle}]$. □

Theorem 3.5. *A function u lies in $\mathrm{Dom}(U_k)$ if and only if the following conditions are satisfied:*

- *$u|_{(t_{j-1}, t_j)} \in H^2((t_{j-1}, t_j))$ for all $j > k$ and*

$$\sum_{j>k} \int_{t_{j-1}}^{t_j} \left(|u''(t)|^2 + |u'(t)|^2 + |u(t)|^2\right) g_k(t)\mathrm{d}t < \infty.$$

- *u and $u' g_k$ are continuos on $[t_k, h_\Gamma)$ and $u(t_k) = 0$.*
- *If $|\Gamma| := \int_\Gamma 1\mathrm{d}t = \int_0^{h_\Gamma} g_\Gamma(t)\mathrm{d}t < \infty$, and hence $h_\Gamma < \infty$, then*

$$\lim_{t \to h_\Gamma} u'(t) g_k(t) = 0.$$

  If $|\Gamma| = \infty$, then

$$\lim_{t \to h_\Gamma} u(t) = 0.$$

On this domain, the operator acts as

$$(U_k u)(t) = -\frac{1}{g_k(t)} \left(g_k(t) u'(t)\right)', \tag{3.27}$$

which clearly means

$$(U_k u)(t) = -u''(t), \quad t \ne t_k, t_{k+1}, \dots \tag{3.28}$$

Proof. [10, Thm.3.4.] □

The equality $\lim_{t \to h_\Gamma} u(t) = 0$ for the case $|\Gamma| = \infty$ can also be derived from the analysis of deficiency indices, carried out in [1].

3.4 The spectrum of Δ

Below $U^{[r]}$ stands for the orthogonal sum of r copies of the self adjoint operator U, and $\sim$ means unitary equivalent.

Theorem 3.6. *Let Γ be a metric tree generated by the sequences $\{t_n\}$ and $\{b_n\}$, then*

$$\Delta \sim U_0 \oplus \sum_{k=1}^{\infty} \oplus U_k^{[b_0 \cdots b_{k-1}(b_k-1)]}. \tag{3.29}$$

By the variational principle, this implies, that the spectral properties of all the operators U_k and of the whole operator Δ are determined by the single operator U_0, which we can see from the following results:

Theorem 3.7. *Let U_k be the above defined operators. Then*

- *If U_0 is positive definite, then the same is true for any operator U_k, $k \in \mathbb{N}$, and*
$$\min\, \sigma(U_0) \le \min\, \sigma(U_1) \le \cdots \le \min\, \sigma(U_k) \le \ldots$$
- *If the spectrum of U_0 is discrete, then the same is true for any operator U_k, $k \in \mathbb{N}$.*
- *If the spectrum of U_0 is discrete, then*
$$\min\, \sigma(U_k) \to \infty \text{ as } k \to \infty.$$

Proof. [10, Thm.3.6.] □

Generally, because of Theorem 3.2., the spectrum of Δ is given by

$$\sigma(\Delta) = \overline{\bigcup_{k=0}^{\infty} \sigma(U_k)}; \quad \sigma_p(\Delta) = \bigcup_{k=0}^{\infty} \sigma_p(U_k). \tag{3.30}$$

This two results together lead to

Corollary 3.8. *On a regular metric tree:*

- *The Laplacian Δ is positive definite if and only if the operator U_0 is positive definite. Moreover,*
$$\min\, \sigma(\Delta) = \min\, \sigma(U_0).$$
- *The spectrum of Δ is discrete if and only if the spectrum of U_0 is discrete.*

We, now want to characterize, whether the spectrum is discrete or not, so we start with the assumption $h_\Gamma < \infty$. In this case we have the following result

Theorem 3.9. *Let Γ be a regular metric tree and $h_\Gamma < \infty$, then the spectrum of Δ is discrete.*

Proof. [10, Thm.4.1.] □

For more information in the case of the discreteness of the spectrum, for example, for the Weyl asymptotic formula for the eigenvalue counting function, see [2], [6], [10] and [11].

So in this case everything is clear, so we can analyze the case, where $h_\Gamma = \infty$. In this case the situation is more delicate. We start by introducing the term L_Γ

$$L_\Gamma = \int_0^{h_\Gamma} \frac{\mathrm{d}t}{g_\Gamma(t)}. \tag{3.31}$$

If $\sup\limits_{\epsilon \in E(\Gamma)} |\epsilon| < \infty$, set $\bar{t} = \sup\limits_{\epsilon \in E(\Gamma)} |\epsilon|$, it follows that $L_\Gamma < \infty$, because

$$\int_0^{h_\Gamma} \frac{\mathrm{d}t}{g_\Gamma} = \sum_{i=1}^{\infty} (t_i - t_{i-1}) \prod_{j=0}^{i} \frac{1}{b_i} \leq \bar{t} \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty$$

because $b_n \geq 2$ for $n \geq 0$. So, the question is, what happens if $\sup\limits_{\epsilon \in E(\Gamma)} |\epsilon| = \infty$, in this case, we have the following theorem:

Theorem 3.10. *Let Γ be a regular metric tree and $\sup\limits_{\epsilon \in E(\Gamma)} |\epsilon| = \infty$, where $|\epsilon|$ is the length of the edge ϵ, then $\sigma(\Delta) = [0, \infty)$.*

Proof. Here, I present the proof of [10, Thm.5.1.]. We will show this result with the help of Weyl sequences, so let $r > 0$ and $\lambda = r^2$, we will proof, that this point belongs to the spectrum. To do this, we fix a non-negative function $\zeta \in C_0^\infty(-1, 1)$ such that $\zeta(t) = 1$ on $(-\frac{1}{2}, \frac{1}{2})$, next, we choose an edge $\epsilon \in E(\Gamma)$. e can be identified with the interval $(-l, l)$, where $l = \frac{|\epsilon|}{2}$ Look at the following function on Γ

$$f(t) = \zeta(t/l) \sin(rt) \quad \text{on } e, \qquad f(t) = 0 \quad \text{otherwise.}$$

This function belongs to the domain of Δ. Now we calculate

$$\left\| \Delta f - r^2 f \right\| = \frac{1}{l} \left\| \frac{\zeta''(t/l)\sin(rt)}{l} + r\zeta'(t/l)\cos(rt) \right\| \leq \frac{1}{l}\text{const.}$$

but the last term goes to zero as l goes to infinitely. So, we can choose a sequence of edges such that $|\epsilon| \to \infty$, to obtain a Weyl sequence for the operator Δ and the point $\lambda = r^2$, which implies $\lambda \in \sigma(\Delta)$ □

Because of the last theorem, we restrict ourselves in the following considerations to the cases were $\sup\limits_{e \in E(\Gamma)} |\epsilon| < \infty$, where clearly, see above $L_\Gamma < \infty$. Next, we continue with a result of positive definiteness of the Laplacian and then with a criterion of discreteness of $\sigma(\Delta)$

Theorem 3.11. *Let Γ be a regular tree and $h_\Gamma = \infty$. Then the Laplacian is positive definite if and only if*

$$B(\Gamma) = B(g_\Gamma) := \sup_{t>0} \left(\int_0^t g_\Gamma(\tau)\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)} \right) < \infty. \tag{3.32}$$

Moreover, $(4B(g_\Gamma))^{-1} \leq \min \sigma(\Delta) \leq B(g_\Gamma)^{-1}$.

Proof. [10, Thm.5.2.] □

It is not hard to give some sequences to satisfy the conditions from above, for example, take $b_n = 2$ for $n \geq 1$ bounded and $t_n - t_{n-1} = 2$ for $n \geq 1$ will do it.

Theorem 3.12. *Let* Γ *be a regular tree and* $h_\Gamma = \infty$*. Then the Laplacian on* Γ *has a discrete spectrum if and only if*

- $B(\Gamma) < \infty$.
- $\lim\limits_{t\to\infty} \left(\int_0^t g_\Gamma(\tau)\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)} \right) = 0.$

Proof. [10, Thm.5.3.] □

So, we have now all tools to characterize nearly all trees. We start our calculations by considering the behaviour of the spectrum by varying the sequence b_n. First of all, we make some definitions to simplify the calculations: Set $a_0 := t_1$ and $a_{n-1} := t_n - t_{n-1}$, then clearly $\sum_{j=0}^{n-1} a_j = t_n$

Theorem 3.13. *Let* $\lim\limits_{n\to\infty} b_n = \infty$ *and* $\inf\limits_{n\in\mathbb{N}} a_n = a > 0$ *then the spectrum of* Δ *is not discrete.*

Proof. Set $G(t) := \int_0^t g_\Gamma(\tau)\mathrm{d}\tau \int_t^\infty \frac{\mathrm{d}\tau}{g_\Gamma(\tau)} > 0$ let $t \in [t_n, t_{n+1}]$, then it follows, that there is an $s \leq a_n$, such that $t_n + s = t$, A straightforward computation shows that

$$G(t) = \left(\sum_{i=0}^{n-1} a_i \prod_{j=0}^{i} \frac{b_j}{b_0 \cdots b_n} + s \right) \left(\sum_{i=n+1}^{\infty} a_i \prod_{j=n+1}^{i} \frac{1}{b_j} + a_n - s \right).$$

We define

$$A_n := \sum_{i=0}^{n-1} a_i \prod_{j=0}^{i} \frac{b_j}{b_0 \cdots b_n}$$

and

$$B_n := \sum_{i=n+1}^{\infty} a_i \prod_{j=n+1}^{i} \frac{1}{b_j}.$$

Since $\lim\limits_{n\to\infty} b_n = \infty$ and $\sup\limits_n a_n < \infty$, it follows that $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} B_n = 0$. Next, we calculate the first derivate, $\frac{\mathrm{d}G(t_n+s)}{\mathrm{d}s}$ of G on (t_n, t_{n+1}). It is given by

$$G'(t_n + s) = -2s + a_n + B_n - A_n.$$

Now, we compute the local extremum of $G(t_n + s)$ in (t_n, t_{n+1}). If there is a local extremum, it is a maximum, because $G(t_n + s)$ is a concave function on (t_n, t_{n+1}). The same reason implies, that there are no other. It is given by

$$s_n = \frac{B_n - A_n + a_n}{2},$$

as long as $s_n \leq a_n$. Since $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} B_n = 0$, there is an n_0, such that $\forall n \geq n_0\ s_n \leq a_n$. Next, we compute

$$\lim_{n\to\infty\ ;\ n\geq n_0} G(t_n + s_n) = \lim_{n\to\infty\ ;\ n\geq n_0} \left(\frac{A_n + B_n + a_n}{2}\right)^2 = \frac{a_n^2}{4} \neq 0$$

Therefore $\lim\limits_{t\to\infty} G(t) > 0$ or do not exist. $\square$

The following corollary is an easy consequence of the last theorem

Corollary 3.14. *Let $\lim\limits_{n\to\infty} b_n = \infty$ and $\lim\limits_{n\to\infty} a_n = 0$ then the spectrum of Δ is discrete.*

We, finally show

Theorem 3.15. *Let $\sup\limits_{n\in\mathbb{N}} b_n < \infty$, therefore, we can set $\max\limits_{n\in\mathbb{N}} b_n = b$ and let $\inf\limits_{n\in\mathbb{N}} a_n > 0$ then the spectrum is not discrete.*

Proof. We start to define $a := \inf\limits_{n\in\mathbb{N}} a_n > 0$.We show that

$$\lim_{n\to\infty} \int_0^{t_n} g_\Gamma(\tau)\mathrm{d}\tau \int_{t_n}^{\infty} \frac{1}{g_\Gamma(\tau)}\mathrm{d}\tau > 0,$$

which implies, that $\lim\limits_{t\to\infty} \int_0^t g_\Gamma(\tau)\mathrm{d}\tau \int_t^\infty \frac{1}{g_\Gamma(\tau)}\mathrm{d}\tau > 0$ or do not exist.

$$\lim_{n\to\infty}\left(\int_0^{t_n} g_\Gamma(\tau)\mathrm{d}\tau \int_{t_n}^{\infty} \frac{\mathrm{d}\tau}{g_\Gamma(\tau)}\right) = \lim_{n\to\infty}\left(\sum_{i=0}^{n} a_i \prod_{j=0}^{i} b_j\right)\left(\sum_{i=n+1}^{\infty} a_i \prod_{j=0}^{i} \frac{1}{b_j}\right)$$

$$= \lim_{n\to\infty}\left(a_n + \sum_{i=0}^{n-1} a_i \prod_{j=i+1}^{n} \frac{1}{b_j}\right)\left(\sum_{i=n+1}^{\infty} a_i \prod_{j=n+1}^{i} \frac{1}{b_j}\right)$$

$$\geq \lim_{n\to\infty} a^2 \left(1 + \sum_{i=1}^{n-1} \frac{1}{b^i}\right)\left(\frac{1}{b}\sum_{i=1}^{\infty}\frac{1}{b^i}\right)$$

$$= \lim_{n\to\infty} a^2 \left(\frac{1-\left(\frac{1}{b}\right)^n}{1-\frac{1}{b}}\right)\frac{1}{b}\left(\frac{1}{1-\frac{1}{b}}\right) = \frac{a^2}{b}\frac{1}{\left(1-\frac{1}{b}\right)^2} > 0,$$

because of $b_j \leq b$ and $a \leq a_j$ for $j \in \mathbb{N}$. $\square$

3.5 AC-Spectrum on a regular tree

In this section, we would like to find the connection between the result of chapter 2 and chapter 3. In chapter 3, we have seen, that

$$\Delta \sim U_0 \oplus \sum_{k=1}^{\infty} \oplus U_k^{[b_0\cdots b_{k-1}(b_k-1)]},$$

where Δ is the Laplace operator on the regular tree and U_k, $k = 0, 1, ...$ are the operators associated with the quadratic form

$$\alpha_k[\phi] = \int_{t_k}^{\infty} |\phi'(t)|^2 g_k(t) \mathrm{d}t.$$

Furthermore, we have seen, that the operators U_k acts as

$$U_k := \frac{1}{g_k}\left(-\frac{\mathrm{d}}{\mathrm{d}x} g_k \frac{\mathrm{d}}{\mathrm{d}x}\right) = \frac{1}{g_\Gamma}\left(-\frac{\mathrm{d}}{\mathrm{d}x} g_\Gamma \frac{\mathrm{d}}{\mathrm{d}x}\right),$$

on the space $L^2((t_k, \infty); g_k(x)\mathrm{d}x)$. So, we will show the following result

Theorem 3.16. *Let Γ be a regular metric tree and Δ be the Laplace operator defined on Γ. Furthermore, let U_k, $k = 0, 1, ...$ be the corresponding operators on the spaces $L^2((t_k, \infty); g_k(x)dx)$. Then, the essential spectrum of all operators U_k, $k = 0, 1, ...$ is the same*

$$\sigma_{ess}(U_0) = \sigma_{ess}(U_k), \text{ for } k \geq 1 \tag{3.33}$$

Moreover, if $\sup_{n\in\mathbb{N}} b_n < \infty$ and $\inf_{n\in\mathbb{N}} a_n > 0$, we can use here Theorem 2.1. Let $u_{k,i}(z; x)$, $i = 1, 2$ be two solutions of $U_k u_{k,i} = z u_{k,i}$, $i = 1, 2$, such that $u_{k,1}(z; t_k) = g_k(t_k) u'_{k,2}(z; t_k) = 0$ and $u_{k,2}(z; t_k) = g_k(t_k) u'_{k,1}(z; t_k) = 1$, $k \geq 0$, $t_0 = 0$ and $g_0(t) = g_\Gamma(t)$, than on

$$S = \left\{E \in \mathbb{R} \mid u_{k,1}\sqrt{g_\Gamma} \text{ and } u_{k,2}\sqrt{g_\Gamma} \text{ are bounded on } [\,0, \infty\,)\right\},$$

the spectrum of U_k and therefore the spectrum of Δ is absolutely continuous.

Proof. First of all remark that the operators U_k, $k \geq 0$ are Sturm Liouville operators, where $V = 0$ and $p = r = g_k$. To see, that the essential spectrum of the operators U_k is equal, remark, that U_0 is nearly the same as U_k, only the Interval $(0, t_k)$ is "cut off". By [12, Thm 9.11], we have the following result

$$\sigma_{ess}(U_0) = \sigma_{ess}(U_0|_{(0,t_k)}) \cup \sigma_{ess}(U_0|_{(t_k,\infty)}).$$

Here $U_0|_I$ has the be understood as the restriction of U_0 to the interval I with a Dirichlet boundary condition at every finite endpoint. Clearly $\sigma_{ess}(U_0|_{(t_k,\infty)}) = \sigma_{ess}(U_k)$. But on $(0, t_k)$, U_0 is a regular Sturm–Liouville operator, which immediately implies, that $\sigma_{ess}(U_0|_{(0,t_k)})$ is empty, by [12, Thm 3.10] and therefore

$$\sigma_{ess}(U_0) = \sigma_{ess}(U_k), \quad k \geq 1.$$

Now let us turn to the second statement. Let $b := \sup_{n\in\mathbb{N}} b_n < \infty$ and $a := \inf_{n\in\mathbb{N}} a_n > 0$. From the construction of $g_k(t)$, $k \geq 0$, it is clear, that

$$g_k(t) \leq e^{\frac{b}{a}t}, \quad k \geq 0.$$

The function on the right hand site is locally bounded and hence $g_k(t)$, $k \geq 0$ are it too. It is easy to see, that both $\gamma(g_\Gamma(t)) < \infty$ and $\omega(g_\Gamma(t), g_\Gamma(t)) < \infty$ and therefore $\gamma(g_k(t)) < \infty$ and $\omega(g_k(t), g_k(t)) < \infty$, because $g_k(t) = (b_0 \cdots b_k)^{-1} g_\Gamma(t)$, for $t \geq t_k$ and we are done. □

Chapter 4

Examples

Here, we want to consider some examples of the Laplace operator on a regular metric tree Γ.

We have not answered the question, whether the spectrum of all operators U_k, given in equation (3.27), is equal, but the answer is no, which can be shown in the following example:

Let $t_n = \sum_{j=0}^{n-1} 2^j$, which clearly means, that $t_n - t_{n-1} = 2^{n-1}$, and $b_n = b$. We restrict our view to the eigenvalues. For U_0, it is easy to see that the eigenvalues are given by $(\pi l)^2$ for $l \in \mathbb{N}$, take therefore the eigenfunctions:

$$(\psi_l(t) = \begin{cases} \sin(\pi l t) & 0 \leq t < t_1 \\ \frac{1}{b}\sin(\pi l t) & t_1 \leq t < t_2 \\ \vdots & \end{cases} .$$

Now we look at U_1, but there we see for example, that the point $(\frac{\pi}{2})^2$ is also in the spectrum. To be exactly, the eigenvalues of the operator U_k are given by $(\frac{\pi l}{2^k})^2$ for $l \in \mathbb{N}$.

Next we want to consider the tree, given by the following sequences:

$$t_n := n \text{ and } b_n := b. \tag{4.1}$$

Because of theorem 3.15, we see that this causes no discrete spectrum, but how can we calculate it. First of all is our tree very regular, all subtrees of the type T_ν for $\nu \in V$ are clearly the same as the whole tree Γ. This is the reason, why all operators are equal und therefore:

$$\Delta \sim U_0^{[\infty]} \tag{4.2}$$

where $U_0^{[\infty]}$ stands for the orthogonal sum of infinitely many copies of the self adjoint operator U_0. So this ensures us that the whole spectral information of Δ is given by the operator U_0, which is defined as in chapter 3, without the perturbation $V(x)$, thanks to theorem 3.16 there, we have to find the regions in $\mathbb{R}$, where the solutions of $U_0 u = Eu$, $E \in \mathbb{R}$ are "bounded", in the sense that

$u\sqrt{g_\Gamma}$ is bounded. We start with rewriting the function g_Γ. In our tree g_Γ is given by

$$g_\Gamma(t) = \begin{cases} b^n & n < t \leq n+1 \\ 1 & t = 0 \end{cases} . \tag{4.3}$$

Next, we know that, both $u(t)$ and $g_\Gamma(t)u'(t)$ have to be continuous on $\mathbb{R}^+$, which means in our example

$$u(n-) = u(n+) \text{ and } u'(n-) = bu'(n+), \qquad \forall n \in \mathbb{N} \tag{4.4}$$

So, we have to search two solutions u_1 and u_2 of $U_0 u = \lambda u$, which we get by taking the solutions of $\Delta u = \lambda u$ and keep in mind the conditions, given in the last equation. This functions should keep $u_1(0) = u_2'(0) = 0$ and $u_1'(0) = u_2(0) = 1$. They are easy to find and given by

$$u_1(x) = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}x) \text{ and } u_2(x) = \cos(\sqrt{\lambda}x). \tag{4.5}$$

So, if we set $x := n + y$, our transfer matrix is given by

$$\widetilde{T}(\lambda; x, 0) = \begin{pmatrix} \cos(\sqrt{\lambda}y) & -\sqrt{\lambda}\sin(\sqrt{\lambda}y) \\ \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}y) & \cos(\sqrt{\lambda}y) \end{pmatrix}$$

$$\left[\begin{pmatrix} 1/b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{\lambda}1) & -\sqrt{\lambda}\sin(\sqrt{\lambda}1) \\ \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}1) & \cos(\sqrt{\lambda}1) \end{pmatrix} \right]^n$$

$$= \begin{pmatrix} \cos(\sqrt{\lambda}y) & -\sqrt{\lambda}\sin(\sqrt{\lambda}y) \\ \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}y) & \cos(\sqrt{\lambda}y) \end{pmatrix} \begin{pmatrix} \frac{1}{b}\cos(\sqrt{\lambda}x) & -\frac{\sqrt{\lambda}}{b}\sin(\sqrt{\lambda}x) \\ \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}x) & \cos(\sqrt{\lambda}x) \end{pmatrix}^n .$$

We want to see, what happens if we go from n to $n+1$, so let

$$\Lambda(\lambda) = \begin{pmatrix} \frac{1}{b}\cos(\sqrt{\lambda}) & -\frac{\sqrt{\lambda}}{b}\sin(\sqrt{\lambda}) \\ \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}) & \cos(\sqrt{\lambda}) \end{pmatrix},$$

then it follows for any solution w, from $U_0 w = \lambda w$, that

$$\begin{pmatrix} w'(n) \\ w(n) \end{pmatrix} = \Lambda(\lambda) \begin{pmatrix} w'(n-1) \\ w(n-1) \end{pmatrix}. \tag{4.6}$$

To proof this, use the last equations and some addition theorems. To know, whether our functions w are bounded, we have to compute the eigenvalues of $\Lambda(\lambda)$. Because we have done a discretization there, we have to regard, when the eigenvalues of Λ are greater or lesser than one. If the eigenvalues are equal to one for some $\lambda \in \mathbb{R}$, we have AC-spectrum there. The Eigenvalues of $\Lambda(\lambda)$ are given by

$$\frac{(1+b)\cos\left(\sqrt{\lambda}\right) \pm \frac{\sqrt{1-6b+b^2+(1+b)^2\cos\left(2\sqrt{\lambda}\right)}}{\sqrt{2}}}{2b}. \tag{4.7}$$

To visualize this, we chose, for example $b = 3$ and get the picture on the next site, where the absolute value of both eigenvalues are marked. The line segments are of the high $\frac{1}{\sqrt{b}}$, in the example $\frac{1}{\sqrt{3}}$. The picture would show us, that there are

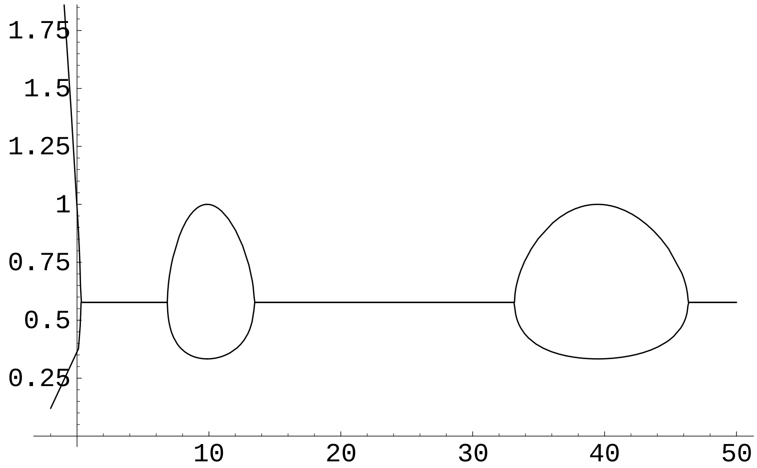

Figure 4.1: Absolute value of both eigenvalues

only eigenvalues, because the Eigenfunctions are exponentially decreasing, but we should remember, that we do not need the eigenfunctions w to be bounded, but wg_Γ, so we have to multiply the matrix $\Lambda(\lambda)$ with $\sqrt{b}$, so we shift the whole spectrum by the factor $\sqrt{b}$. Then, the line segments are in the high of one and therefore, in this regions, the Eigenfunctions are bounded and hence we have there AC-Spectrum. To compute the starts and the ends of the gaps, we have to solve

$$1 - 6b + b^2 + (1+b)^2 \cos\left(2\sqrt{\lambda}\right) = 0. \tag{4.8}$$

The solution is given by

$$\lambda = \left(\arccos\left(\frac{2}{b^{1/2} + b^{-1/2}}\right)\right)^2 =: \theta^2.$$

Because of the arccos, the bands are given by

$$\left[(\pi(l-1)+\theta)^2, (\pi l - \theta)^2\right] \qquad l \in \mathbb{N} \tag{4.9}$$

We see, if we let grow b, the bands of the AC-spectrum are going to shrink more and more, besides, if we define Ξ_n as the spectrum of the Laplacian for $b = n$, it is clear that

$$\Xi_1 \supset \Xi_2 \supset \Xi_3 \supset \ldots . \tag{4.10}$$

Up to this, we already have not said anything about the Eigenvalues. But they can easily be shown. Let $\lambda_l = (\pi l)^2$ for $l \in \mathbb{N}$ and set

$$\psi_l(t) = \begin{cases} \sin(\pi l t) & 0 \le t < 1 \\ \frac{1}{b}\sin(\pi l t) & 1 \le t < 2 \\ & \vdots \end{cases} .$$

Next we define $C := \int_0^1 \sin(\pi l t)^2 \mathrm{d}t < \infty$. The function $\psi_l(t)$ keeps all conditions and is square integrable, because

$$\int_0^\infty \psi_l(t)^2 g_\Gamma(t)\mathrm{d}t = \sum_{j=0}^\infty \frac{b^j}{b^{2j}} \int_0^1 \sin(\pi l t)^2 \mathrm{d}t = \frac{1}{1+1/b} C < \infty.$$

So, we have found the Eigenvalues $\lambda_l = (\pi l)^2$ for $l \in \mathbb{N}$.

We can generalize the last example; we take the length of the edges arbitrary but equal, realized by the sequences

$$b_n = b \text{ and } t_n = tn$$

where $t \in \mathbb{R}^+$ is arbitrary. The results are analogue:
The matrix $\Lambda(\lambda)$ is changing to

$$\begin{pmatrix} u'(tn) \\ u(tn) \end{pmatrix} = \Lambda(\lambda) \begin{pmatrix} u'(t(n-1)) \\ u(t(n-1)) \end{pmatrix}$$

where

$$\Lambda(\lambda) = \begin{pmatrix} \frac{1}{b}\cos(t\sqrt{\lambda}) & -\frac{\sqrt{\lambda}}{b}\sin(t\sqrt{\lambda}) \\ \frac{1}{\sqrt{\lambda}}\sin(t\sqrt{\lambda}) & \cos(t\sqrt{\lambda}) \end{pmatrix}$$

So, we have to find again the eigenvalues of this matrix, but they can easily be found and the AC-spectrum-bands are in this case given by

$$\left[\left(\frac{\pi(l-1)+\theta}{t}\right)^2, \left(\frac{\pi l-\theta}{t}\right)^2\right] \qquad l \in \mathbb{N} \tag{4.11}$$

and theta is defined as above. The eigenvalues and the eigenfunctions are given by

$$(\frac{\pi l}{t})^2 \qquad l \in \mathbb{N}, \text{ and } \psi_l(x) = \begin{cases} \sin(\frac{\pi l x}{t}) & 0 \le x < t \\ \frac{1}{b}\sin(\frac{\pi l x}{t}) & t \le x < 2t \\ & \vdots \end{cases} \tag{4.12}$$

So, we see, that the length of the edges shifts and jolts or stretches the spectrum.

Appendix A

Notation

L^1	the space of all integrable functions
L^2	the space of all square integrable functions
$L^1_{loc}(I)$	the set of all functions $I \to \mathbb{C}$, which are locally integrable
$AC_{loc}(I)$	the set of all functions $I \to \mathbb{C}$, which are locally absolutely continuous
$W(u, v)$	the Wronskian evaluated on x
τ	the Sturm-Liouville operator
H	the Schrödinger operator
$\mathfrak{D}(\cdot)$	the domain
$\mathbb{C}$	the complex numbers
$\mathbb{R}$	the real numbers
C^1	the room of continuously differentiable functions
Γ	a rooted metric tree
E	the set of edges on Γ
V	the set of vertices on Γ
g_Γ	the branching function on Γ
$M(\Gamma)$	the linear space of measurable functions on Γ, that are finite almost everywhere
$M_c(\Gamma)$	the space of all functions in $M(\Gamma)$, supported by only finitely many edges
$\mathfrak{M}_\Gamma$	the subspace of $M(\Gamma)$, consisting of all symmetric functions on Γ
$\sigma(\cdot)$	the spectrum
$\sigma_p(\cdot)$	the point spectrum
z^*	the complex conjugate of z

Bibliography

[1] R. Carlson, Nonclassical Sturm–Liouville problems and Schrödinger operators on radial trees, Electron. J. Differential Equations **2000**, No. 71, 24 pp.

[2] W.D. Evans, D.J. Harris and J. Lang, *The approximation numbers of Hardy type operators on trees*, Proc. London Math. Soc. (3) **83** (2001), no. 2, 390–418.

[3] D.J. Gilbert, *On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints*, Proc. Roy. Soc. Edinburgh Sect. A **112** (1989), no. 3-4, 213–229.

[4] D.J. Gilbert and D.B. Pearson, *On subordinacy and analysis of the spectrum of one dimensional Schrödinger operators*, J. Math. Anal. Appl. **128** (1987), no. 1, 30–56.

[5] S.Khan and D.B. Pearson, *Subordinacy and spectral theory for infinite matrices*, Helv. Phys. Acta, 505–527, 1992.

[6] N. Naimark and M. Solomyak, *Eigenvalue Estimates for the weighted Laplacian an metric trees*, Proc. London Math. Soc. (3) **80** (2000), no. 3, 690–724.

[7] N. Naimark and M. Solomyak *Geometry of Sobolev spaces on regular trees and the Hardy Inequalities*, Russ. J. Math. Phys. **8** (2001), no. 3, 322–335.

[8] D.B. Pearson, *Quantum Scattering and Spectral Theory. Techniques of Physics*, Academic Press, London, 1988.

[9] B. Simon, *Bounded eigenfunctions and absolutely continuous spectra for one–dimensional Schrödinger operators*, Proc. Amer. Math. Soc. **124** (1996), no 11, 3361–3369 .

[10] M. Solomyak *On the spectrum of the Laplacian on regular metric trees*, Waves Random Media **14** (2004), no. 1, S155–S171.

[11] M. Solomyak, *Laplace and Schrödinger operators on regular metric trees: The discrete spectrum case*, in *Function spaces, differential operators and nonlinear analysis (Teistungen, 2001)*, 161–181, Birkhäuser, Basel, 2003.

[12] G. Teschl, *Mathematical Methods in Quantum Mechanics*, lecture notes.

Curriculum Vitae

I was born on August 2, 1984, in Vienna, where I went to primary school and afterwards to a grammar school with the main focus on musical education. In June 2002, I received my A-levels. Then I started to study Mathematics at the University of Vienna.