universität wien

MASTERARBEIT

Titel der Masterarbeit

The Korteweg–de Vries Equation: Long-Time Asymptotics in the Similarity Region

Verfasser

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angestrebter akademischer Grad

Master of Science (MSc)

Wien, im Juni 2016

Studienkennzahl lt. Studienblatt: A 066 821
Studienrichtung lt. Studienblatt: Masterstudium Mathematik
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Abstract

The Korteweg–de Vries equation is a nonlinear partial differential equation used to describe the propagation of shallow water waves. The objective of the present thesis consists in determining the long-time behavior of its solutions in the similarity region. This will be achieved through a combination of results from scattering theory with the method of nonlinear steepest descent method for oscillatory Riemann–Hilbert problems. While this approach is in principal well known, the focus here will be on several technical aspects which were previously not addressed in full detail.

Zusammenfassung

Die Korteweg-de-Vries Gleichung ist eine nichtlineare, partielle Differentialgleichung, die verwendet wird, um die Ausbreitung von Flachwasserwellen zu beschreiben. Das Ziel dieser Arbeit besteht darin, das Verhalten ihrer Lösungen für große Zeiten in der Similaritätsregion zu bestimmen. Dies wird durch eine Kombination von Resultaten aus der Streutheorie mit der Methode des nichtlinearen, steilsten Abstiegs für oszillierende Riemann-Hilbert-Probleme erreicht. Diese Herangehensweise ist grundsätzlich bereits bekannt, doch liegt der Fokus hier auf einigen technischen Aspekten, die zuvor nicht im Detail behandelt wurden.

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CHAPTER 1

Introduction

In the mathematical field of wave theory, several nonlinear partial differential equations are investigated in order to gain insight into the physical phenomenon of waves. A particularly interesting one is the so-called Korteweg–de Vries equation given by

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \qquad (x,t) \in \mathbb{R} \times \mathbb{R},$$

where the subscripts x and t are indicating differentiation with respect to the corresponding variable. It is mostly used to model the behavior of waves in shallow water. Its history goes back to the year of 1834 (see [**9**]), when Scott Russell witnessed an unusual wave spreading in a canal and started conducting experiments to understand his observation. This was followed by more theoretical investigations through Boussinesq and Lord Rayleigh in the 1870s. However, the equation is named after Korteweg and de Vries, who published their considerations in 1895.

A central problem appearing in the context of the equation is to determine the long-time asymptotics for its solutions. Here, it suffices to understand the case $t \to \infty$, since for any solution $q(x,t)$ the function $q(-x,-t)$ is a solution as well. If the solution is well-behaved in the sense of existence of certain moments, the method of inverse scattering theory provides an elegant approach to its investigation. More precisely, the solution is uniquely determined by its so-called spectral data consisting of the reflection coefficient $R(k,t)$, a finite number of values $0 < \kappa_1 < ... < \kappa_N$ and norming constants $\gamma_1(t)$,...,$\gamma_N(t)$.
In determining the long-times asymptotics, one seeks to find an asymptotic expansion in terms of the spectral data. The main observation is that eventually, the solution will consist of a a number of solitons moving to the right and a small radiation part moving to the left. A more diligent analysis shows, that there are four main regions to distinguish:

(i) The Soliton Region:
If $x/t > C$ for some $C > 0$, then

$$q(x,t) \sim -2 \sum_{j=1}^{N} \frac{\kappa_j^2}{\cosh^2(\kappa_j x - 4\kappa_j^3 t - p_j)},$$

where the phase shift p_j equals

$$p_j = \frac{1}{2} \log \left(\frac{\gamma_j(0)^2}{2\kappa_j} \prod_{l=j+1}^{N} \left(\frac{\kappa_l - \kappa_j}{\kappa_l + \kappa_j} \right)^2 \right).$$

Note that each term in the above sum represents a wave traveling to the right. This is a classical result and was proven for example in Grunert and Teschl [**12**].

(ii) The Self-Similar Region:
If $|x/(3t)^{1/3}| \leq C$ for some $C > 0$, the solution is connected to the Painléve II transcendent. Further information on this can be found in Segur and Ablowitz [**21**].

(iii) The Collisionless Shock Region:
This region is given by $x < 0$ and $C^{-1} < -x/(3t\log(t)^2)^{1/3} < C$ for some $C > 1$. It only shows in the generic case of $R(0,0) = -1$ and was discussed for example in Deift, Venakides and Zhou [**7**].

(iv) The Similarity Region:
If $x/t < -C$ for some $C > 0$, then

$$q(x,t) \sim \left(\frac{4\nu(k_0)k_0}{3t}\right)^{1/2} \sin\left(16tk_0^3 - \nu(k_0)\log(192tk_0^3) + \delta(k_0)\right), \tag{1.1}$$

where

$$k_0 = \sqrt{-\frac{x}{12t}}$$

denotes the stationary phase point and

$$\begin{aligned}
\nu(k_0) &= -\frac{1}{2\pi}\log\left(1 - |R(k_0,0)|^2\right), \\
\delta(k_0) &= \frac{\pi}{4} - \arg\left(R(k_0,0)\right) + \arg\left(\Gamma(\mathrm{i}\nu(k_0))\right) + 4\sum_{j=1}^{N}\arctan\left(\frac{\kappa_j}{k_0}\right) \\
&\qquad - \frac{1}{\pi}\int_{-k_0}^{k_0}\log\left(\frac{1-|R(\zeta,0)|^2}{1-|R(k_0,0)|^2}\right)\frac{d\zeta}{\zeta-k_0}.
\end{aligned}$$

This result will be obtained in Theorem 5.6. The similarity region is also investigated in Ablowitz and Segur [**1**] and Buslaev and Sukhanov [**5**].

Here, we focus on the similarity region only and compute the long-time asymptotics under certain growth and analycity asssumptions (see Chapter 2). The result will be proven by applying the nonlinear steepest descent method for oscillatory Riemann–Hilbert problems. This technique was introduced by Deift and Zhou [**8**] and originally employed in the context of the modified Korteweg–de Vries equation. In essence, this thesis is a detailed and rigorous discussion of parts of [**12**]. However, some notable and necessary changes have been made to the presentation there. A thorough analysis of the proof of Theorem 5.2 in [**12**] reveals a small error, which originates from the fact that the estimates in Theorem 5.1 are valid for $\zeta > \rho_0$, but not $\zeta \geq \rho_0$. Therefore, a new and improved version of Theorem 5.1 is presented here (see Theorem 4.1). Its proof is slightly more elaborate and involves properties of the Cauchy operator on the H^1 rather than the L^2 space. Moreover, the final step in the proof of Theorem 5.4 in [**12**] is not fully justified by the previous results obtained there. This issue has been solved here via introducing an expansion for the respective error term in Lemma 5.5. Last but not least, the original proof for uniqueness of solutions for vector-valued problems has been replaced by a more elementary one adopted from [**3**].

This thesis is organized as follows:
Chapter 2 contains relevant facts from the field of scattering theory and a precise formulation of our assumptions on the solution of the Korteweg–de Vries equation. Moreover, we introduce a vector Riemann–Hilbert problem connected to the spectral data and point out the link between the solution of this problem and the solution of the equation.
In **Chapter 3**, this Riemann–Hilbert problem is replaced by another one more suitable for further investigation.
Chapter 4 discusses asymptotic expansions for solutions of matrix Riemann–Hilbert problems. In essence, this is done via approximating certain well-behaved problems by explicitly solvable ones. Via the fundamental connection between Riemann–Hilbert problems and singular integral equations we then obtain the main result of this chapter.
Finally, in **Chapter 5** we prove the above long-time asymptotics in the similarity region. Roughly speaking, we recall the jump matrix of the transformed vector problem in Chapter 3 and consider the associated matrix problem. The findings of Chapter 4 give the asymptotics of the matrix solution. Using this result, we can obtain the asymptotics for the vector solution of the transformed problem. Then we know the asymptotics for the solution of the original vector problem on a large set and can use the results in Chapter 2 to translate them into properties of the solution of the Korteweg–de Vries equation.

Acknowledgments

First of all, I would like to thank my supervisor Gerald Teschl, who I am indebted to, for his continual academic support and guidance throughout the project. His various suggestions were an enormous help and I am grateful for many enlightening discussions.

I would also like to extend my thanks to my parents, who provided me with financial support throughout my studies and taught me how much difference passion about your work can make, and to my wonderful sister, who took the time to listen to convergence criteria for infinite sums although not understanding anything.

Furthermore, I wish to thank my colleagues at university - it has been a beautiful experience discovering mathematics with them. I am deeply grateful to all my friends for their support, especially to Teresa Ruetz and Katharina Fröch, who were there for me at all times.

Finally, my most special thanks go to Christof Ender for being my cup of coffee in need.

CHAPTER 2

The Riemann–Hilbert problem associated to the equation

The goal of this thesis is to gain insight into the asymptotic behavior of a solution $q(x,t)$ of the Korteweg–de Vries equation for large values of t. This will be achieved via the methods of scattering theory. The basic idea behind this procedure is informally summarized in the following. Assuming a Schrödinger operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + v \tag{2.1}$$

with a potential v, there are sets S of spectral data that uniquely determine v. Typical data contained in such a set consist of the spectrum of the operator, special eigenvectors and related objects. Now, suppose $q(x,t)$ satisfies the Korteweg–de Vries equation and the initial condition $q(x,0) = q_0(x)$. Then one may consider for every $t \in \mathbb{R}$ the one-dimensional Schrödinger operator

$$\mathcal{L}(t) = -\frac{d^2}{dx^2} + q(\cdot,t). \tag{2.2}$$

It turns out that for a certain set of spectral data S, the time evolution of the spectral data $S(t)$ of the operator $\mathcal{L}(t)$ is extremely simple. More precisely, the objects in $S(t)$ are subject to ordinary differential equations that can be explicitly solved. If the initial data $q_0(x)$ is known, so is the spectral data set $S(0)$ and thus also $S(t)$ for every $t \in \mathbb{R}$. It then suffices to apply an inverse scattering transform to recover $q(x,t)$ from the spectral data. In essence, one thus replaces the complicated law governing the time evolution of $q(x,t)$ by the simple one for $S(t)$, i.e. the Korteweg–de Vries equation by a system of ordinary differential equations.

In this thesis, we focus only on the final inverse transform step and recover the solution $q(x,t)$ asymptotically from the spectral data via a Riemann–Hilbert problem. The neccessary results concerning the other steps will be provided without proof in Section 2.1. Namely, we specify the spectral data set S and state its time evolution. Section 2.2 then introduces the Riemann–Hilbert problem through which the recovering is performed. Throughout this thesis, we will always suppose $q(x,t)$ is a fixed real-valued, classical and decaying solution of the Korteweg–de Vries equation, where the latter means

$$\max_{|t|\le T} \int_{\mathbb{R}} (1+|x|) \left(|q(x,t)| + \sum_{j=1}^{3} \left| \frac{\partial^j q}{\partial x^j}(x,t) \right| \right) dx < \infty, \quad \text{for all } T > 0. \tag{2.3}$$

2.1. Scattering theory

The information provided in this section has essentially been collected from [**17**] and [**6**].

Assume a real-valued potential $v(x)$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} (1+|x|)|v(x)|\, dx < \infty,$$

and consider the Sturm-Liouville equation

$$-u''(x) + u(x)v(x) = k^2 u(x), \qquad x \in \mathbb{R} \tag{2.4}$$

for k in the closed upper half plane. Then there is a construction involving an integral kernel (see [**17**], Chapter 3), that gives two special solutions $\psi^{\pm}(k,x)$ of this equation. These solutions are called the Jost solutions of the equation and have the following properties:

(i) The Jost solutions asymptically look like the solutions $\mathrm{e}^{\pm \mathrm{i}kx}$ of the Sturm-Liouville equation with $v \equiv 0$, meaning

$$\lim_{k\to\pm\infty} \mathrm{e}^{\mp \mathrm{i}kx}\psi^{\pm}(k,x) = 1$$

for every $k \in \mathbb{C}$, $\mathrm{Im}(k) \geq 0$.

(ii) In the following, we denote by $W(f,g)(x) = f'(x)g(x) - f(x)g'(x)$ the Wronskian of two differentiable functions f and g in the point x. Assume that $k \in \mathbb{C}$ and $f(x)$, $g(x)$ both are solutions to the Sturm-Liouville equation (2.4) with parameter k. Then their Wronskian $W(f,g)(x)$ is constant on $\mathbb{R}$.
Considering the particular case of $k \in \mathbb{R}$, $k \neq 0$ and the Jost solutions, it turns out

$$W\left(\psi^+(k,\cdot),\ \psi^+(-k,\cdot)\right)(x) = 2\mathrm{i}k = W\left(\psi^-(-k,\cdot),\ \psi^-(k,\cdot)\right)(x)$$

for every $x \in \mathbb{R}$. Consequently, the pairs of functions $(\psi^+(k,x),\ \psi^+(-k,x))$ and $(\psi^-(-k,x),\ \psi^-(k,x))$ are fundamental systems of the Sturm-Liouville equation (2.4) for $k \in \mathbb{R}$, $k \neq 0$.

(iii) There is a conjugate relation in form of $\psi^{\pm}(-k,x) = \overline{\psi^{\pm}(k,x)}$, $k \in \mathbb{R}$.

(iv) The Jost solutions $\psi^{\pm}(k,x)$ are continuous for $\mathrm{Im}(k) \geq 0$ and analytic for $\mathrm{Im}(k) > 0$.

For $k \in \mathbb{R}$, $k \neq 0$, the fundamental system property implies

$$\psi^+(k,x) = b(k)\psi^-(k,x) + a(k)\psi^-(-k,x), \tag{2.5}$$

$$\psi^-(k,x) = -b(-k)\psi^+(k,x) + a(k)\psi^+(-k,x), \tag{2.6}$$

where

$$a(k) = \frac{W\left(\psi^+(k,\cdot),\ \psi^-(k,\cdot)\right)}{W\left(\psi^-(-k,\cdot),\ \psi^-(k,\cdot)\right)} \qquad \text{and} \qquad b(k) = \frac{W\left(\psi^-(-k,\cdot),\ \psi^+(k,\cdot)\right)}{W\left(\psi^-(-k,\cdot),\ \psi^-(k,\cdot)\right)}.$$

A further investigation shows, that $a(k)$ is analytic in the upper half plane and has only finitely many zeroes there. These zeroes lie on the imaginary axis and coincide with the points on the upper half plane, where $\psi^+(k,x)$ and $\psi^-(k,x)$ are linearly dependent. In the following, we denote them by $\mathrm{i}\kappa_1, ..., \mathrm{i}\kappa_N$, where

$0 < \kappa_1 < ... < \kappa_N$. Furthermore, we let $\mu_j^{\pm}$ be the unique constants such that $\psi^{\pm}(\mathrm{i}\kappa_j, x) = \mu_j^{\pm}\psi^{\mp}(\mathrm{i}\kappa_j, x)$ and define the left and right norming constants by

$$\gamma_j^+ = \left\|\psi^+(\mathrm{i}\kappa_j,\cdot)\right\|_{L^2(\mathbb{R})}^{-1} \qquad \text{and} \qquad \gamma_j^- = \left\|\psi^-(\mathrm{i}\kappa_j,\cdot)\right\|_{L^2(\mathbb{R})}^{-1}.$$

In this notation, $\mathrm{i}a'(\mathrm{i}\kappa_j) = (\mu_j^-)^{-1}(\gamma_j^-)^{-2} = (\mu_j^+)^{-1}(\gamma_j^+)^{-2}$ and so the zeroes $\mathrm{i}\kappa_j$ are simple. Finally, we introduce the left and right reflection coefficients $R^{\pm}(k)$ and the transmission coefficient $T(k)$ by setting

$$R^+(k) = -\frac{b(-k)}{a(k)}, \quad R^-(k) = \frac{b(k)}{a(k)}, \qquad T(k) = \frac{1}{a(k)}, \qquad k \in \mathbb{R}\backslash\{0\}.$$

Recent results (see [**10**]) show that the above conditions on the potential $v(x)$ imply

$$R^{\pm}(k),\ T(k) - 1 \in \mathcal{A},$$

where $\mathcal{A}$ denotes the Wiener algebra consisting of the Fourier transforms of functions in $L^1(\mathbb{R})$. Since all functions in $\mathcal{A}$ are continuous, a meaning can be given to the values of $R^{\pm}(k)$ and $T(k)$ at zero by continuous extension. The origin is the only real number, at which the reflection coefficients possibly take values outside the open unit disc, since

$$|R^+(k)|,\ |R^-(k)| \ < 1, \qquad k \in \mathbb{R}\backslash\{0\}.$$

However, the transmission coefficient is tending to 1 for large values of k and we even have

$$T(k) = 1 + O\left(\frac{1}{k}\right), \qquad \text{as } |k| \to \infty,\ \mathrm{Im}(k) \geq 0.$$

Obviously, $R^{\pm}(k)$ and $T(k)$ inherit the conjugation property from the Jost solutions, meaning

$$R^+(-k) = \overline{R^+(k)}, \quad R^-(-k) = \overline{R^-(k)}, \qquad T(-k) = \overline{T(k)}, \qquad k \in \mathbb{R}.$$

Finally, we have collected enough spectral data to recover the potential v. If $S^+(\mathcal{L}) = \{R^+(k), (\kappa_j, \gamma_j^+); 1 \leq j \leq N\}$ and $S^-(\mathcal{L}) = \{R^-(k), (\kappa_j, \gamma_j^-); 1 \leq j \leq N\}$, then each of the collections $S_+(\mathcal{L})$ and $S_-(\mathcal{L})$ uniquely determines v. The data sets $S_+(\mathcal{L})$ and $S_-(\mathcal{L})$ are called the right and left scattering data for the Schrödinger operator $\mathcal{L}$ in equation (2.1). Moreover, the right data can be obtained from the left data and vice versa, if we combine that $\log(|T(\cdot)|^2) \in L^1(\mathbb{R})$ with the formulas $-a'(\mathrm{i}\kappa_j)^2 = (\gamma_j^-)^{-2}(\gamma_j^+)^{-2}$,

$$T(k) = \prod_{j=1}^{N} \frac{k + \mathrm{i}\kappa_j}{k - \mathrm{i}\kappa_j} \quad \mathrm{e}^{\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} \frac{\log(|T(\zeta)|^2)}{\zeta - k}\, d\zeta}, \qquad k \in \mathbb{C},\ \mathrm{Im}(k) > 0, \tag{2.7}$$

and

$$|T(k)|^2 + |R^{\pm}(k)|^2 = 1, \qquad T(k)\overline{R^+(k)} + \overline{T(k)}R^-(k) = 0, \qquad k \in \mathbb{R}. \tag{2.8}$$

Now returning to our fixed real-valued and rapidly decreasing solution $q(x,t)$ of the Korteweg–de Vries equation, the above theory applies to $q(\cdot,t)$ for every fixed $t \in \mathbb{R}$. We then write $\psi^{\pm}(k,x,t)$ for the Jost solutions of the operator $\mathcal{L}(t)$ in equation (2.2) and use the analogous notation for the other spectral data introduced above. Our aim is to determine the time evolution of these objects. It turns out, that the transmission coefficient $T(k,t) = T(k)$ is in fact time-independent, so (2.5) and (2.6) can be rewritten in form of the scattering relations

$$T(k)\psi^{\mp}(k,x,t) = \overline{\psi^{\pm}(k,x,t)} + R^{\pm}(k,t)\psi^{\pm}(k,x,t), \qquad k,x,t \in \mathbb{R}. \tag{2.9}$$

The other relevant properties of $T(k)$ are summarized in the next lemma.

LEMMA 2.1. *The transmission coefficient $T(k)$ is meromorphic on the upper half plane $\{\mathrm{Im}(z) > 0\}$ with a finite number of poles $\mathrm{i}\kappa_1,\dots,\mathrm{i}\kappa_N$, $0 < \kappa_1 < \dots < \kappa_N$ and continuous up to the real line without the origin. The poles are simple and the residues satisfy*

$$\mathrm{Res}_{\mathrm{i}\kappa_j} T(k) = \mathrm{i}\mu_j^+(t)\gamma_j^+(t)^2 = \mathrm{i}\mu_j^+(0)\gamma_j^+(0)^2, \qquad t \in \mathbb{R}.$$

Furthermore, the functions $T(k)$ and $k/T(k)$ are bounded for $\mathrm{Im}(k) \geq 0$, $k \neq 0$ close to the origin.

In view of the above argument, $q(\cdot,t)$ is uniquely determined by both $S^+(t)$ and $S^-(t)$. We choose to use the right scattering data for the reconstruction and set $R(k,t) = R^+(k,t)$, $\gamma_j(t) = \gamma_j^+(t)$ in order to simplify the notation. For the data contained in $S(0)$, we additionally write

$$R(k) := R^+(k,0) \text{ and } \gamma_j := \gamma_j^+(0).$$

The next lemma finally illustrates the time evolution of $S^+(t)$.

LEMMA 2.2. *For $t \in \mathbb{R}$, the right reflection coefficient $R(k,t)$ and the right norming constants $\gamma_j(t)$ are given by*

$$R(k,t) = R(k)\mathrm{e}^{8\mathrm{i}k^3 t} \quad \textit{and} \quad \gamma_j(t) = \gamma_j \mathrm{e}^{4\kappa_j^3 t}.$$

In many situations, it turns out essential to have precise results concerning the growth rate of the reflection coefficient $R(k)$. Under the above assumptions on the solution $q(x,t)$ of the Korteweg–de Vries equation (see Theorem 1 in [**6**]),

$$R(k) = O\left(\frac{1}{k^4}\right), \qquad \text{for } |k| \to \infty,\ k \in \mathbb{R}.$$

However, in order to to provide a streamlined and elegant approach, even extra analyticity and boundedness conditions will be imposed on $R(k)$ and $T(k)$ in this thesis. These assumptions can then be weakened using analytic approximation (see for example [**12**]). From now on, we will rely on the below two hypothesis.

HYPOTHESIS 2.3. *There exists some small $\delta_R > 0$ such that the following conditions are satisfied:*

(i) The reflection coefficient $R(k)$ can be extended to a holomorphic and bounded function on the strip $-\delta_R < \mathrm{Im}(k) < \delta_R$.

(ii) The extension of the reflection coefficient has order

$$R(k) = O\left(\frac{1}{k}\right),$$

for $|k| \to \infty$, $-\delta_R < \mathrm{Im}(k) < \delta_R$.

Under these assumptions, the transmission coefficient $T(k)$ can be extended holomorphically to the strip $-\delta_R < \mathrm{Im}(k) < \delta_R$ as well. In fact, we may define $T(k)$ in the lower half plane by setting

$$T(k) := \frac{1 - R(k)R(-k)}{T(-k)} \qquad \text{for } -\delta_R < \mathrm{Im}(k) < 0.$$

Then $T(k)$ is holomorphic in the negative strip $-\delta_R < \mathrm{Im}(k) < 0$ and continuous for $|\mathrm{Im}(k)| < \delta_R$, $k \neq 0$, since

$$\lim_{k\to x,\ \mathrm{Im}(k)<0} T(k) = \frac{1 - R(x)R(-x)}{T(-x)} = T(x) \qquad \text{for } x \in \mathbb{R}\backslash\{0\}.$$

By Morera's Theorem, $T(k)$ is holomorphic in $\{|\mathrm{Im}(k)| < \delta_R\}\backslash\{0\}$. Applying Lemma 2.1, there is a constant C such that $|T(k)| \leq C/|k|$ for $k \neq 0$ close to the origin. But this implies that the singularity at zero cannot not essential. As $|T(k)| \leq 1$ on the real axis, it is removable and $T(k)$ can be extended holomorphically to $\{|\mathrm{Im}(k)| < \delta_R\}$.

Hypothesis 2.4. *The extension $T(k)$ of the transmission coefficient vanishes nowhere except possibly the origin, i.e.*

$$T(k) \neq 0 \qquad\qquad \text{for } k \neq 0,\ |\mathrm{Im}(k)| < \delta_R.$$

Remark 2.5. *Using a similar reasoning as in Theorem 4.1 of* [**11**], *one can show that these two hypothesis are satisfied if*

$$\int_{\mathbb{R}} |q(x,0)|\, e^{\delta_R |x|}\, dx < \infty.$$

2.2. The Riemann–Hilbert problem

In this section, we use the scattering data to set up a meromorphic vector Riemann–Hilbert problem and show that the solution $q(x,t)$ of the Korteweg–de Vries equation can be recovered from the solution of the Riemann–Hilbert problem.

Theorem 2.6. *Suppose, $\{R(k),\ (\kappa_j, \gamma_j);\ 1 \leq j \leq N\}$ is the right scattering data of $\mathcal{L}(0)$. For $x, t \in \mathbb{R}$ define the phase $\Phi(\cdot, x, t)$ by*

$$\Phi(k,x,t) = 8\mathrm{i}k^3 + 2\mathrm{i}k\frac{x}{t}, \qquad k \in \mathbb{C}, \tag{2.10}$$

and consider the following meromorphic vector Riemann–Hilbert problem:
Find $m\colon \mathbb{C}\backslash(\mathbb{R} \cup \{\pm\mathrm{i}\kappa_j;\ j = 1, ..., N\}) \to \mathbb{C}^2$ such that:

- *(i) The first component m_1 is holomorphic on $\mathbb{C}\backslash(\mathbb{R} \cup \{\mathrm{i}\kappa_j;\ j = 1, ..., N\})$ and has simple poles at the points $\mathrm{i}\kappa_j$. The second component m_2 is holomorphic on $\mathbb{C}\backslash(\mathbb{R} \cup \{-\mathrm{i}\kappa_j;\ j = 1, ..., N\})$ and has simple poles at the points $-\mathrm{i}\kappa_j$. Moreover, the residues satisfy the pole condition*

$$\begin{aligned} \mathrm{Res}_{\mathrm{i}\kappa_j}\, m_1(k) &= \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} m_2(\mathrm{i}\kappa_j), \\ \mathrm{Res}_{-\mathrm{i}\kappa_j}\, m_2(k) &= -\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} m_1(-\mathrm{i}\kappa_j) \end{aligned} \tag{2.11}$$

(ii) There exist continuous extensions $m_\pm$ of m from the punctured half planes $\{z \in \mathbb{C};\ \pm\mathrm{Im}(z) > 0\}\backslash\{\pm\mathrm{i}\kappa_j; j = 1, ..., N\}$ to $\mathbb{R}$ and the jump condition

$$m_+(k) = m_-(k)v(k), \qquad v(k) = \begin{pmatrix} 1-|R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-t\Phi(k)} \\ R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}, \tag{2.12}$$

holds for $k \in \mathbb{R}$.

(iii) m satisfies the symmetry condition

$$m(-k) = m(k)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.13}$$

(iv) The behavior of m near infinity is given by the normalization

$$m(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} + O\left(\frac{1}{k}\right), \qquad |k| \to \infty. \tag{2.14}$$

Then, for all $x, t \in \mathbb{R}$ the above problem has a unique solution $m(\cdot, x, t)$ given by

(2.15)

$$m(k,x,t) := \begin{cases} \begin{pmatrix} T(k)\psi^-(k,x,t)\mathrm{e}^{\mathrm{i}kx} & \psi^+(k,x,t)\mathrm{e}^{-\mathrm{i}kx} \end{pmatrix}, & \mathrm{Im}(k) > 0, \\ \begin{pmatrix} \psi^+(-k,x,t)\mathrm{e}^{\mathrm{i}kx} & T(-k)\psi^-(-k,x,t)\mathrm{e}^{-\mathrm{i}kx} \end{pmatrix}, & \mathrm{Im}(k) < 0. \end{cases}$$

Proof. We start by showing that $m(k,x,t)$ solves the Riemann–Hilbert problem. The symmetry condition (ii) is obvious from the defition. In view of Lemma 2.1, the assertions concerning the holomorphicity hold true and

$$\begin{aligned} \mathrm{Res}_{\mathrm{i}\kappa_j}\, m_1(k,x,t) &= \psi^-(\mathrm{i}\kappa_j,x,t)\mathrm{e}^{\mathrm{i}^2\kappa_j x}\,\mathrm{Res}_{\mathrm{i}\kappa_j}\, T(k) = \psi^-(\mathrm{i}\kappa_j,x,t)\mathrm{e}^{\mathrm{i}^2\kappa_j x}\mathrm{i}\mu_j^+(t)\gamma_j(t)^2 \\ &= \mathrm{i}\gamma_j^2\mathrm{e}^{8\kappa_j^3 t}\mathrm{e}^{\mathrm{i}^2\kappa_j x}\psi^+(\mathrm{i}\kappa_j,x,t) = \mathrm{i}\gamma_j^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}m_2(\mathrm{i}\kappa_j,x,t). \end{aligned}$$

If we use the symmetry condition (ii), this also implies

$$\begin{aligned} \mathrm{Res}_{-\mathrm{i}\kappa_j}\, m_2(k) &= \lim_{k\to-\mathrm{i}\kappa_j}(k+\mathrm{i}\kappa_j)m_2(k,x,t) = -\lim_{k\to\mathrm{i}\kappa_j}(k-\mathrm{i}\kappa_j)m_2(-k,x,t) \\ &= -\mathrm{Res}_{\mathrm{i}\kappa_j}\, m_1(k,x,t) = -\mathrm{i}\gamma_j^2\mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}m_1(-\mathrm{i}\kappa_j,x,t). \end{aligned}$$

Next, we turn to verifying the jump condition. By Lemma 2.1 and the properties of the Jost solutions, $m(k,x,t)$ can be extended continuously from the left and right to the real line. Using the conjugation property for $T(k)$, $R^\pm(k,t)$ and $\psi^\pm(k,x,t)$, the time evolution of the reflection coefficient and the scattering relations (2.9) and (2.8), we obtain

$$\begin{aligned} (m_-(k,x,t)v(k))_1 &= |T(k)|^2\psi^+(-k,x,t)\mathrm{e}^{\mathrm{i}kx} + R(k)\mathrm{e}^{t\Phi(k)}T(-k)\psi^-(-k,x,t)\mathrm{e}^{-\mathrm{i}kx} \\ &= \left(|T(k)|^2\psi^+(-k,x,t) + R(k,t)T(-k)\psi^-(-k,x,t)\right)\mathrm{e}^{\mathrm{i}kx} \\ &= \left(|T(k)|^2\psi^+(-k,x,t) - T(k)\overline{R^-(k,t)}\psi^-(-k,x,t)\right)\mathrm{e}^{\mathrm{i}kx} \\ &= \left(|T(k)|^2\psi^+(-k,x,t) - T(k)R^-(-k,t)\psi^-(-k,x,t)\right)\mathrm{e}^{\mathrm{i}kx} \\ &= \left(|T(k)|^2\psi^+(-k,x,t) - T(k)(T(-k)\psi^+(-k,x,t) - \overline{\psi^-(-k,x,t)})\right)\mathrm{e}^{\mathrm{i}kx} \\ &= m_+(k,x,t)_1 \end{aligned}$$

for $k \in \mathbb{R}$. Similarly, we have

$$\begin{aligned} (m_-(k,x,t)v(k))_2 &= -R(-k)\mathrm{e}^{-t\Phi(k)}\psi^+(-k,x,t)\mathrm{e}^{\mathrm{i}kx} + T(-k)\psi^-(-k,x,t)\mathrm{e}^{-\mathrm{i}kx} \\ &= \left(-R(-k,t)\psi^+(-k,x,t) + T(-k)\psi^-(-k,x,t)\right)\mathrm{e}^{-\mathrm{i}kx} \end{aligned}$$

$$= \overline{\psi^+(-k,x,t)}\mathrm{e}^{-\mathrm{i}kx} = m_+(k,x,t).$$

Finally, the normalization condition will follow immediately from Lemma 2.7.

It remains to prove uniqueness of solutions. Suppose $m, \tilde{m}$ are solutions to the above problem. Then by linearity $\hat{m}(k) := m(k) - \tilde{m}(k)$ satisfies the jump condition (ii), the symmetry relation (iii) and a new normalization condition of the type $\hat{m}(k) = O(1/k), |k| \to \infty$. Now, $\hat{m}_1$ is holomorphic on $\mathbb{C}\backslash(\mathbb{R} \cup \{\mathrm{i}\kappa_j;\ j = 1, ..., N\})$ and at a point $\mathrm{i}\kappa_j$ we find either a simple pole or a removable singularity. The same holds for $\hat{m}_2$ with the points $\mathrm{i}\kappa_j$ replaced by $-\mathrm{i}\kappa_j$. Applying condition (i) for m and $\tilde{m}$ now yields

$$\begin{aligned} \lim_{k \to \mathrm{i}\kappa_j} (k - \mathrm{i}\kappa_j)\hat{m}_1(k) &= \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}\hat{m}_2(\mathrm{i}\kappa_j), \\ \lim_{k \to -\mathrm{i}\kappa_j} (k + \mathrm{i}\kappa_j)\hat{m}_2(k) &= -\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}\hat{m}_1(-\mathrm{i}\kappa_j). \end{aligned} \tag{2.16}$$

Next we define an auxiliary scalar-valued function

$$F(k) := \hat{m}_1(k)\overline{\hat{m}_1(\overline{k})} + \hat{m}_2(k)\overline{\hat{m}_2(\overline{k})}$$

on the punctured upper half plane $\{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}\backslash\{\mathrm{i}\kappa_j;\ j = 1, ..., N\}$ and translate the obtained properties of $\hat{m}$ into properties of F. Since for a holomorphic function $g(z)$, the function $\overline{g(\overline{z})}$ is holomorphic, too, $F(k)$ is holomorphic on its domain. As above, a point $\mathrm{i}\kappa_j$ is either a removable singularity or a simple pole of F. Using symmetry (iii) for $\hat{m}$ and the previous limit calculations, we find

$$\begin{aligned} \lim_{k \to \mathrm{i}\kappa_j} (k - \mathrm{i}\kappa_j)F(k) &= \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}\hat{m}_2(\mathrm{i}\kappa_j)\overline{\hat{m}_1(-\mathrm{i}\kappa_j)} + \hat{m}_2(\mathrm{i}\kappa_j)\overline{\lim_{k \to \mathrm{i}\kappa_j}(\overline{k} + \mathrm{i}\kappa_j)\hat{m}_2(\overline{k})} \\ &= \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}\hat{m}_2(\mathrm{i}\kappa_j)\overline{\hat{m}_1(-\mathrm{i}\kappa_j)} + \hat{m}_2(\mathrm{i}\kappa_j)\overline{-\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}\hat{m}_1(-\mathrm{i}\kappa_j)} \\ &= 2\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}\hat{m}_2(\mathrm{i}\kappa_j)\overline{\hat{m}_1(-\mathrm{i}\kappa_j)} \\ &= 2\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}|\hat{m}_2(\mathrm{i}\kappa_j)|^2. \end{aligned}$$

Note that this limit is zero, if the point $\mathrm{i}\kappa_j$ is a removable singularity of F, and equal to the residue, if the point $\mathrm{i}\kappa_j$ is a simple pole. The above normalization condition for $\hat{m}$ leads to $F(k) = O(k^{-2})$ for $|k| \to \infty$ and the jump condition (ii) for $\hat{m}$ gives that F can be extended continuously to the real line by

$$\begin{aligned} F_+ &= \hat{m}_{1,+}\overline{\hat{m}_{1,-}} + \hat{m}_{2,+}\overline{\hat{m}_{2,-}} \\ &= (\hat{m}_{1,-}(1 - |R|^2) + \hat{m}_{2,-}R\mathrm{e}^{t\Phi})\overline{\hat{m}_{1,-}} + (\hat{m}_{2,-} - \hat{m}_{1,-}\overline{R}\mathrm{e}^{-t\Phi})\overline{\hat{m}_{2,-}} \\ &= |\hat{m}_{1,-}|^2(1 - |R|^2) + \hat{m}_{2,-}\overline{\hat{m}_{1,-}}R\mathrm{e}^{t\Phi} + |\hat{m}_{2,-}|^2 - \overline{\hat{m}_{2,-}}\hat{m}_{1,-}\overline{R}\mathrm{e}^{-t\Phi} \\ &= (1 - |R|^2)|\hat{m}_{1,-}|^2 + |\hat{m}_{2,-}|^2 + 2\mathrm{i}\,\mathrm{Im}(\overline{\hat{m}_{1,-}}\hat{m}_{2,-}R\mathrm{e}^{t\Phi}). \end{aligned}$$

Finally, we will use a contour integration method to conclude that $\hat{m} \equiv 0$. Indeed, if we orient the half circle

$$\gamma_\rho = [-\rho, \rho] \cup \{\rho \mathrm{e}^{i\theta};\ 0 \le \theta \le \pi\}$$

contourclockwise, Cauchy's residue theorem yields

$$\int_{\gamma_\rho} F(k)\, dk = 2\pi\mathrm{i} \sum_{j=1}^N \lim_{k \to \mathrm{i}\kappa_j}(k - \mathrm{i}\kappa_j)F(k) = -4\pi \sum_{j=1}^N \gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}|\hat{m}_2(\mathrm{i}\kappa_j)|^2$$

for every $\rho > \kappa_N$. Now letting ρ tend to infinity, the integral over the circle part of γ_ρ clearly tends to zero due to the behavior of F at infinity. So we are left with

$$\int_{\mathbb{R}} F_+(k)\, dk + 4\pi \sum_{j=1}^{N} \gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} |\hat{m}_2(\mathrm{i}\kappa_j)|^2 = 0.$$

Noting that $\Phi(\mathrm{i}\kappa_j) \in \mathbb{R}$, taking the real part on both sides implies

$$\int_{\mathbb{R}} (1 - |R(k)|^2)|\hat{m}_{1,-}(k)|^2 + |\hat{m}_{2,-}(k)|^2\, dk + 4\pi \sum_{j=1}^{N} \gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} |\hat{m}_2(\mathrm{i}\kappa_j)|^2 = 0.$$

Since $|R(k)| < 1$ for $k \in \mathbb{R}\backslash\{0\}$ and $\gamma_j, \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)} > 0$, we can conclude that

$$\hat{m}_2(\mathrm{i}\kappa_1) = ... = \hat{m}_2(\mathrm{i}\kappa_N) = 0, \qquad \hat{m}_{1,-}(k) = \hat{m}_{2,-}(k) = 0, \quad k \in \mathbb{R}.$$

Remembering (2.16), the first equation implies that $\mathrm{i}\kappa_j$ is a removable singularity of $\hat{m}_1$ and the symmetry condition (iii) for $\tilde{m}$ gives the corresponding statement for $\hat{m}_2$ and the point $-\mathrm{i}\kappa_j$. By the above and the jump condition (ii) for $\tilde{m}$ we have $\hat{m}_+ = \hat{m}_- \equiv 0$ on $\mathbb{R}$, so all in all $\hat{m}$ is holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and continuously extendable to the whole of $\mathbb{C}$. By Morera's theorem, $\hat{m}$ is entire . Finally, Liouville's theorem combined with the normalization condition for $\hat{m}$ leads to $\hat{m} \equiv 0$. □

Finally, we provide a method to recover the solution of the Korteweg–de Vries equation from the solution of the meromorphic vector Riemann–Hilbert problem.

Lemma 2.7. *Let $(x,t) \in \mathbb{R} \times \mathbb{R}$. Then*

$$m_1(k,x,t)\, m_2(k,x,t) = 1 + \frac{q(x,t)}{2k^2} + o\left(\frac{1}{k^2}\right), \qquad \text{as } |k| \to \infty,$$

and

$$m(k,x,t) = \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{Q(x,t)}{2\mathrm{i}k} \begin{pmatrix} -1 & 1 \end{pmatrix} + O\left(\frac{1}{k^2}\right), \qquad \text{as } |k| \to \infty,$$

where $Q(x,t) := -\int_x^\infty q(y,t)\, dy$.

Proof. Fix $(x,t) \in \mathbb{R} \times \mathbb{R}$. The Jost solutions and the transmission coefficient have asymptotic expansions given by (see Lemma 9 in [**6**])

$$\mathrm{e}^{-\mathrm{i}kx}\psi^+(k,x,t) = 1 + \frac{1}{2\mathrm{i}k} Q_+(x,t) + \frac{1}{2(2\mathrm{i}k)^2} Q_+(x,t)^2 - \frac{q(x,t)}{(2\mathrm{i}k)^2} + o\left(\frac{1}{k^2}\right),$$

$$\mathrm{e}^{\mathrm{i}kx}\psi^-(k,x,t) = 1 + \frac{1}{2\mathrm{i}k} Q_-(x,t) + \frac{1}{2(2\mathrm{i}k)^2} Q_-(x,t)^2 - \frac{q(x,t)}{(2\mathrm{i}k)^2} + o\left(\frac{1}{k^2}\right),$$

$$T(k) = 1 + \frac{1}{2\mathrm{i}k} \int_{-\infty}^{\infty} q(y,t)\, dy + \frac{1}{2(2\mathrm{i}k)^2} \left(\int_{-\infty}^{\infty} q(y,t)\, dy\right)^2 + o\left(\frac{1}{k^2}\right),$$

for $|k| \to \infty$, $\mathrm{Im}(k) \ge 0$, with $Q_+(x,t) = Q(x,t)$ and $Q_-(x,t) = -\int_{-\infty}^{x} q(y,t)\, dy$. With this result, the above claim can easily be verified.

□

CHAPTER 3

Transformation of the Riemann–Hilbert problem

In this chapter, we will replace the meromorphic Riemann–Hilbert problem for a pair $(x,t) \in \mathbb{R}\times\mathbb{R}$ by a holomorphic one with jump matrices that are in a certain sense close to the identity, if t is large. The transformation will consist of substituting the pole conditions by additional jump conditions, a conjugation step and finally contour deformation.

Throughout this chapter, it is assumed (x,t) is a fixed pair in $\mathbb{R}\times\mathbb{R}$. Also, from now on, if a function f is considered that can be extended continuously from the left resp. right to a contour Γ, f_+ resp. f_- will denote the extension function from the left resp. right restricted to Γ.

3.1. Replacing pole conditions by holomorphic jump conditions

The following lemma shows that the given meromorphic Riemann–Hilbert problem is equivalent to a holomorphic one.

LEMMA 3.1. *Assume $\epsilon > 0$ is such that the circles $\{|k\pm \mathrm{i}\kappa_j| = \epsilon\}$ are disjoint and do not intersect the real line. Orient the circle around $\mathrm{i}\kappa_j$ counterclockwise and the one around $-\mathrm{i}\kappa_j$ clockwise. Suppose further $m\colon \mathbb{C}\backslash(\mathbb{R}\cup\{\pm\mathrm{i}\kappa_j;\ j=1,...,N\}) \to \mathbb{C}^2$ and define*

$$
\check{m}(k) := \begin{cases} m(k)\begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 1 \end{pmatrix}, & |k-\mathrm{i}\kappa_j| < \epsilon, \\ m(k)\begin{pmatrix} 1 & \frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k+\mathrm{i}\kappa_j} \\ 0 & 1 \end{pmatrix}, & |k+\mathrm{i}\kappa_j| < \epsilon, \\ m(k), & \text{else}. \end{cases} \tag{3.1}
$$

Then m solves the meromorphic Riemann–Hilbert problem in Theorem 2.6 if and only if $\check{m}$ solves the following holomorphic Riemann–Hilbert problem:
Find $\check{m}\colon \mathbb{C}\backslash(\mathbb{R}\cup\bigcup_{j=1}^N\{|k-\mathrm{i}\kappa_j|=\epsilon\}\cup\{|k+\mathrm{i}\kappa_j|=\epsilon\}) \to \mathbb{C}^2$ such that:
Both components of $\check{m}$ are holomorphic, the jump condition (2.12) on $\mathbb{R}$, the symmetry relation (2.13) and the normalization (2.14) from Theorem 2.6 hold true, there is a continuous extension of $\check{m}$ from the left (resp. right) to the circle around $\pm\mathrm{i}\kappa_j$ and the extensions satisfy

$$
\begin{aligned} \check{m}_+(k) &= \check{m}_-(k)\begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 1 \end{pmatrix}, & |k-\mathrm{i}\kappa_j| = \epsilon, \\ \check{m}_+(k) &= \check{m}_-(k)\begin{pmatrix} 1 & -\frac{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}{k+\mathrm{i}\kappa_j} \\ 0 & 1 \end{pmatrix}, & |k+\mathrm{i}\kappa_j| = \epsilon. \end{aligned} \tag{3.2}
$$

PROOF. We only show that $\check{m}$ solves the holomorphic problem, the other direction being similar. Since

$$\lim_{k\to \mathrm{i}\kappa_j}(k-\mathrm{i}\kappa_j)\check{m}_1(k) = \lim_{k\to \mathrm{i}\kappa_j}(k-\mathrm{i}\kappa_j)m_1(k) - \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}m_2(k) = 0,$$

$\check{m}$ is holomorphically extendable to the disc around $\mathrm{i}\kappa_j$. On the circle around $\mathrm{i}\kappa_j$, we have $\check{m}_- = (m_1, m_2)$ and $\check{m}_+ = (m_1 - \mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}/(k-\mathrm{i}\kappa_j)m_2, m_2)$, which leads to (3.2). The calculation for the circle around $-\mathrm{i}\kappa_j$ is analogous, and (2.13) is also easily verified. Finally, (2.12) and (2.14) clearly remain valid for $\check{m}$. □

Now we define $\check{m}(k)$ to be the function that is obtained from $m(k)$ by equation (3.1) with $\epsilon = \check{\epsilon}$, where

$$\check{\epsilon} := \frac{1}{4}\min\{\kappa_1,\ \kappa_2-\kappa_1,\ ...,\ \kappa_N-\kappa_{N-1}\}.$$

The previous lemma then tells us, that instead of $m(k)$, we can equivalently investigate $\check{m}(k)$.

3.2. The partial transmission coefficient

This section is devoted to introducing the partial transmission coefficient and some of its properties.
Our ultimate goal is to reduce the new holomorphic Riemann–Hilbert problem for (x,t) to a problem with a jump matrix that is somehow close to the identity for t large. The definition of $\check{v}$ thus suggests to consider the sign of $\mathrm{Re}(\Phi(k))$. Setting

$$k_0 := \sqrt{-\frac{x}{12t}}\ >\ 0,$$

we find that k_0 and $-k_0$ are the two stationary points of $\Phi(k)$, i.e. the zeros of $\Phi'(k)$. The situation can then be described by Figure 3.1.

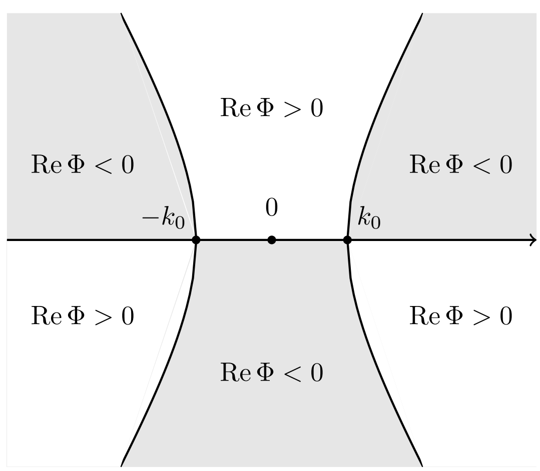


FIGURE 3.1. Sign of $\mathrm{Re}\,\Phi$ for $x/t = -12$.

If we investigate $\check{v}$ on the small circles around the points $\pm\mathrm{i}\kappa_j$, we see that $|\mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}|$ is large for t large. The following lemma indicates how we will later move this term to the denominator.

Lemma 3.2. *Let $\kappa > 0$, $a \in \mathbb{C}\backslash\{0\}$ and $\epsilon > 0$ so small that the circles with radius ϵ around the points $\mathrm{i}\kappa$ and $-\mathrm{i}\kappa$ are disjoint. Orient the circle around $\mathrm{i}\kappa$ counterclockwise and the one around $-\mathrm{i}\kappa$ clockwise. Suppose further that M is a continuous, $\mathbb{C}^2$-valued function defined on $U\backslash(\{|k-\mathrm{i}\kappa| = \epsilon\} \cup \{|k+\mathrm{i}\kappa| = \epsilon\}$ for some open neighborhood U of the union of the closed discs $\overline{D(\mathrm{i}\kappa,\epsilon)} \cup \overline{D(-\mathrm{i}\kappa,\epsilon)}$ and set*

$$\tilde{M}(k) = M(k)D_{\kappa,\epsilon,a}(k), \qquad k \in U\backslash(\{|k-\mathrm{i}\kappa| = \epsilon\} \cup \{|k+\mathrm{i}\kappa| = \epsilon\},$$

where

$$D_{\kappa,\epsilon,a}(k) = \begin{cases} \begin{pmatrix} \frac{k-\mathrm{i}\kappa}{k+\mathrm{i}\kappa} & \frac{k+\mathrm{i}\kappa}{a} \\ -\frac{a}{k+\mathrm{i}\kappa} & 0 \end{pmatrix}, & |k-\mathrm{i}\kappa| < \epsilon, \\ \begin{pmatrix} 0 & \frac{a}{k-\mathrm{i}\kappa} \\ -\frac{k-\mathrm{i}\kappa}{a} & \frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa} \end{pmatrix}, & |k+\mathrm{i}\kappa| < \epsilon, \\ \begin{pmatrix} \frac{k-\mathrm{i}\kappa}{k+\mathrm{i}\kappa} & 0 \\ 0 & \frac{k+\mathrm{i}\kappa}{k-\mathrm{i}\kappa} \end{pmatrix}, & \text{else.} \end{cases}$$

Then M is holomorphic if and only if $\tilde{M}$ is holomorphic and M can be extended continuously from the left resp. right to $\{|k \pm \mathrm{i}\kappa| = \epsilon\}$ if and only if $\tilde{M}$ can be extended. In this case, M satisfies the jump conditions

$$M_+(k) = M_-(k)\begin{pmatrix} 1 & 0 \\ \frac{a}{k-\mathrm{i}\kappa} & 1 \end{pmatrix}, \qquad |k-\mathrm{i}\kappa| = \epsilon,$$
$$M_+(k) = M_-(k)\begin{pmatrix} 1 & \frac{a}{k+\mathrm{i}\kappa} \\ 0 & 1 \end{pmatrix}, \qquad |k+\mathrm{i}\kappa| = \epsilon,$$

if and only if $\tilde{M}$ satisfies

$$\tilde{M}_+(k) = \tilde{M}_-(k)\begin{pmatrix} 1 & \frac{(k+\mathrm{i}\kappa)^2}{a(k-\mathrm{i}\kappa)} \\ 0 & 1 \end{pmatrix}, \qquad |k-\mathrm{i}\kappa| = \epsilon,$$
$$\tilde{M}_+(k) = \tilde{M}_-(k)\begin{pmatrix} 1 & 0 \\ \frac{(k-\mathrm{i}\kappa)^2}{a(k+\mathrm{i}\kappa)} & 1 \end{pmatrix}, \qquad |k+\mathrm{i}\kappa| = \epsilon.$$

If U is symmetric (i.e., $U = -U$), the symmetry condition (2.13) is equivalently satisfied for M and $\tilde{M}$. If U is unbounded, M and $\tilde{M}$ are asymptotically equivalent at infinity, meaning

$$\tilde{M}(k) = M(k)(\mathbb{I} + o(1)), \qquad \text{as } |k| \to \infty,\ k \in U.$$

Proof. The statements concerning holomorphicity and continuous extendability are valid, since $D_{\kappa,\epsilon,\alpha}$ is holomorphic and continuously extendable from both sides to the circles. For symmetry, note that

$$D_{\kappa,\epsilon,\alpha}(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D_{\kappa,\epsilon,\alpha}(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The asymptotic relation is clear, as $D_{\kappa,\epsilon,\alpha}(k) \to \mathbb{I}$ for $|k| \to \infty$. Finally, assume continuous extendability is satisfied. Let v and $\tilde{v}$ be the jump matrices of M and

$\tilde{M}$. Then we find $\tilde{v} = D_{\kappa,\epsilon,\alpha,-}^{-1} v D_{\kappa,\epsilon,\alpha,+}$ on the circles. The lemma thus follows by a direct calculation. □

Now turning to the jump on $\mathbb{R}$, we start by giving an observation hopefully serving as a motivation for our later steps. If we assume a decomposition $\check{v} = v_r v_l$ where v_l (resp. v_r) is a continuous extension of a holomorphic matrix on a strip $0 < \mathrm{Im}(k) < 2\delta$ (resp. $-2\delta < \mathrm{Im}(k) < 0$), we may redefine the vector solution $\check{m}(k)$ by $\hat{m}(k) := \check{m}(k) v_l(k)^{-1}$ for $0 < \mathrm{Im}(k) < \delta$ and by $\hat{m}(k) := \check{m}(k) v_r(k)$ for $-\delta < \mathrm{Im}(k) < 0$. Then $\hat{m}$ will have no jump on $\mathbb{R}$ and be discontinuous along $\mathrm{i}\delta + \mathbb{R}$ and $-\mathrm{i}\delta + \mathbb{R}$ with jump matrices v_l and v_r. If v_l and v_r are triangular matrices with diagonal terms equal to 1 and small off-diagonal terms, this is exactly what we want. Unfortunately, there is no such decomposition. In fact, the LU factorization

$$v(k) = b_r(k)^{-1} b_l(k), \quad b_l(k) = \begin{pmatrix} 1 & 0 \\ R(k)\mathrm{e}^{t\Phi(k)} & 1 \end{pmatrix}, \quad b_r(k) = \begin{pmatrix} 1 & \overline{R(k)}\mathrm{e}^{-t\Phi(k)} \\ 0 & 1 \end{pmatrix},$$

for $k \in \mathbb{R}$ will not have the desired property, if $\mathrm{Re}(k)$ is close to $[-k_0, k_0]$. This naturally suggests using a different factorization of $\check{v}$ in this region. Opposite to the first one, it should contain $\mathrm{e}^{-t\Phi}$ in the left and $\mathrm{e}^{t\Phi}$ in the right matrix. Indeed, there is an LDU factorization given by

$$v(k) = B_r(k)^{-1} \begin{pmatrix} 1 - |R(k)|^2 & 0 \\ 0 & \frac{1}{1-|R(k)|^2} \end{pmatrix} B_l(k), \qquad k \in \mathbb{R}\backslash\{0\},$$

where

$$B_l(k) = \begin{pmatrix} 1 & -\frac{\overline{R(k)}\mathrm{e}^{-t\Phi(k)}}{1-|R(k)|^2} \\ 0 & 1 \end{pmatrix}, \quad B_r(k) = \begin{pmatrix} 1 & 0 \\ -\frac{R(k)\mathrm{e}^{t\Phi(k)}}{1-|R(k)|^2} & 1 \end{pmatrix}.$$

The key idea now is that the Riemann–Hilbert problem corresponding to the unwanted diagonal matrix can be explicitly solved using the theory for scalar Riemann–Hilbert problems. This will lead us to a problem with a jump matrix that has two decompositions with the desired properties.

Inspired by Lemma 3.2 and the above ideas, we define the partial transmission coefficient with respect to k_0 by

$$T(k, k_0) := \prod_{j=1}^{N} \frac{k + \mathrm{i}\kappa_j}{k - \mathrm{i}\kappa_j} \quad \mathrm{e}^{\frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \frac{\log(|T(\zeta)|^2)}{\zeta - k} d\zeta}$$

for $k \in \mathbb{C}\backslash([-k_0, k_0] \cup \{\mathrm{i}\kappa_1, \ldots \mathrm{i}\kappa_N\})$. As already stated in Section 2.1, our assumptions on the potential imply $\log(|T(\cdot)|^2) \in L^1(\mathbb{R})$ and

$$T(k) = \prod_{j=1}^{N} \frac{k + \mathrm{i}\kappa_j}{k - \mathrm{i}\kappa_j} \quad \mathrm{e}^{\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{\log(|T(\zeta)|^2)}{\zeta - k} d\zeta}, \qquad k \in \mathbb{C},\ \mathrm{Im}(k) > 0. \tag{3.3}$$

This motivates the terminology and proves that $T(k, k_0)$ is well-defined, although the integrand has a singularity at zero in the case that $|R(0)| = 1$. The main properties of $T(k, k_0)$ are summarized in the next theorem.

Theorem 3.3. *The partial transmission coefficient is a meromorphic function on* $\mathbb{C}\backslash[-k_0, k_0]$ *with simple poles at the points* $\mathrm{i}\kappa_j$ *and simple zeroes at the points*

$-\mathrm{i}\kappa_j$. *It can be extended continuously from the left and right to the open intervals* $(-k_0, 0)$ *and* $(0, k_0)$. *The extensions satisfy the jump condition*

$$T_+(k,k_0) = (1-|R(k)|^2)\, T_-(k,k_0), \qquad k \in (-k_0,0)\cup(0,k_0). \tag{3.4}$$

On the upper half plane, $T(k,k_0)$ *can be represented as*

$$T(k,k_0) = T(k)\mathrm{e}^{\eta(k,k_0)}, \qquad \mathrm{Im}(k)>0,\ k\neq \mathrm{i}\kappa_j, \tag{3.5}$$

where $\eta(k,k_0)$ *is a holomorphic function on* $\mathbb{C}\backslash((-\infty,-k_0]\cup[k_0,\infty))$. *The behavior at infinity is given by*

$$T(k,k_0) = 1 + \mathrm{i}\, T_1(k_0)\frac{1}{k} + O\left(\frac{1}{k^2}\right), \qquad \text{as } |k|\to\infty, \tag{3.6}$$

where

$$T_1(k_0) = \sum_{j=1}^N 2\kappa_j + \frac{1}{2\pi}\int_{-k_0}^{k_0} \log(|T(\zeta)|^2)\, d\zeta.$$

Moreover,

$$T(-k,k_0) = T(k,k_0)^{-1} = \overline{T(\overline{k},k_0)}, \qquad k\in\mathbb{C}\backslash[-k_0,k_0]. \tag{3.7}$$

Proof. The meromorphicity and the statement concerning the poles and zeroes is clear. To prove continuity from left and right on $(0,k_0)$, let $\epsilon>0$. Then we write

$$\frac{1}{2\pi\mathrm{i}}\int_{-k_0}^{k_0}\frac{\log(|T(\zeta)|^2)}{\zeta-k}\,d\zeta = \frac{1}{2\pi\mathrm{i}}\int_{-k_0}^{\epsilon}\frac{\log(|T(\zeta)|^2)}{\zeta-k}\,d\zeta + \frac{1}{2\pi\mathrm{i}}\int_{\epsilon}^{k_0}\frac{\log(|T(\zeta)|^2)}{\zeta-k}\,d\zeta$$

for $k\in\mathbb{C}\backslash[-k_0,k_0]$. The first integral defines a holomorphic function on $\mathbb{C}\backslash[-k_0,\epsilon]$. The second one is the Cauchy integral $\mathcal{C}\phi(k)$ for the function $\varphi(k) = \log(1 - R(k)\overline{R(k)})$ on $[\epsilon,k_0]$. By our assumptions on $R(k)$, $\varphi(k)$ is Lipschitz continuous on $[\epsilon,k_0]$. Theorem A.1 implies, that $\mathcal{C}\phi(k)$ can be extended continuously from the left and right to (ϵ,k_0) and $(\mathcal{C}\phi)_+ - (\mathcal{C}\phi)_- = \varphi$. Therefore, $T(k,k_0)$ can be extended as well and the jump condition holds for $\epsilon<k<k_0$. This proves the claim, as $\epsilon>0$ was arbitrary.

For $\mathrm{Im}(k)>0$, $k\neq\mathrm{i}\kappa_j$, we can use (3.3) to get

$$\frac{T(k)}{T(k,k_0)} = \mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}\backslash[-k_0,k_0]}\frac{\log(|T(\zeta)|^2)}{\zeta-k}\,d\zeta},$$

which gives (3.5). To deduce (3.6), define the auxiliary function $f(z) := T(1/z,k_0)$. Since $\lim_{z\to 0} f(z) = 0$, f is holomorphically extendable in 0 by Riemann's Theorem. Now an easy calculation yields

$$f'(0) = \lim_{z\to0} f'(z) = \lim_{z\to 0} T'(1/z,k_0)(-1/z^2) = \mathrm{i}\, T_1(k_0).$$

This leads to the asymptotics (3.6), if we set $z=1/k$ in the power series expansion of f. Using $T(-\zeta) = \overline{T(\zeta)}$ for $\zeta\in\mathbb{R}$, we finally find

$$\begin{aligned} T(-k,k_0) &= \prod_{j=1}^N \frac{-k+\mathrm{i}\kappa_k}{-k-\mathrm{i}\kappa_j}\mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_{-k_0}^{k_0}\frac{\log(|T(\zeta)|^2)}{\zeta+k}\,d\zeta} \\ &= \prod_{j=1}^N \frac{k-\mathrm{i}\kappa_j}{k+\mathrm{i}\kappa_j}\mathrm{e}^{\frac{1}{2\pi\mathrm{i}}\int_{-k_0}^{k_0}\frac{\log(|T(-\zeta)|^2)}{-\zeta+k}\,d\zeta} = T(k,k_0)^{-1}. \end{aligned}$$

The second equality in (3.7) can be proven similarly. □

The following investigation of the the partial transmission coefficient near the singularities $\pm k_0$ will turn out helpful later on.

LEMMA 3.4. *The partial transmission coefficient* $T(k, k_0)$ *can be represented as*

(3.8)
$$T(k, k_0) = \left(\frac{k-k_0}{k+k_0}\right)^{\mathrm{i}\nu} \prod_{j=1}^{N} \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j}\, \mathrm{e}^{\psi(k,k_0)}, \quad k \in \mathbb{C}\backslash([-k_0, k_0] \cup \{\mathrm{i}\kappa_1, ..., \mathrm{i}\kappa_N\}),$$

where the branch of the logarithm on $\mathbb{C}\backslash[0,\infty)$ *with* $\arg(k) \in (-\pi, \pi)$ *is used to define the power,* $\nu = -\frac{1}{2\pi}\log(|T(k_0)|^2)$ *and*

$$\psi(k, k_0) = \frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right) \frac{d\zeta}{\zeta - k}, \qquad k \in \mathbb{C}\backslash[-k_0, k_0].$$

The exponent $\psi(k, k_0)$ *is holomorphic on* $\mathbb{C}\backslash[-k_0, k_0]$ *and can be continuously extended to* $\mathbb{C}\backslash(-k_0, k_0)$. *The two integrand functions* $\log(|T(\zeta)|^2/|T(k_0)|^2)/(\zeta - k_0)$ *and* $\log(|T(\zeta)|^2/|T(k_0)|^2)/(\zeta + k_0)$ *are in* $L^1((-k_0, k_0))$ *and*

$$\psi(k_0, k_0) = \frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right) \frac{d\zeta}{\zeta - k_0},$$
$$\psi(-k_0, k_0) = \frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0} \log\left(\frac{|T(\zeta)|^2}{|T(k_0)|^2}\right) \frac{d\zeta}{\zeta + k_0}.$$

In particular, $\mathrm{Re}(\psi(k_0, k_0)) = 0$ *and* $\mathrm{Re}(\psi(-k_0, k_0)) = 0$.

PROOF. To obtain the decomposition, first notice that for $k \in \mathbb{C}\backslash[-k_0, k_0]$ the logarithm term is well-defined, as

$$\frac{k-k_0}{k+k_0} = \frac{(k-k_0)(\overline{k}+k_0)}{|k+k_0|^2} = \frac{|k|^2 - k_0^2}{|k+k_0|^2} + 2\mathrm{i}k_0 \frac{\mathrm{Im}(k)}{|k+k_0|^2} \quad \notin [0, \infty).$$

By the identity theorem, it suffices to verify (3.8) for $k > k_0$. But here we have

$$\int_{-k_0}^{k_0} \frac{1}{\zeta - k}\, d\zeta = -\int_{k-k_0}^{k+k_0} \frac{1}{s}\, ds = \log(k-k_0) - \log(k+k_0) = \log\left(\frac{k-k_0}{k+k_0}\right).$$

Next, we investigate ψ near k_0 and write

$$\psi(k, k_0) = \frac{1}{2\pi\mathrm{i}} \int_{-k_0}^{k_0/2} \frac{\varphi(\zeta)}{\zeta - k}\, d\zeta + \frac{1}{2\pi\mathrm{i}} \int_{k_0/2}^{k_0} \frac{\varphi(\zeta)}{\zeta - k}\, d\zeta = h(k) + c(k),$$

where $\varphi(\zeta) = \log(|T(\zeta)|^2/|T(k_0)|^2)$. Then $h(k)$ is holomorphic on $\mathbb{C}\backslash[-k_0, k_0/2]$. Also, $\varphi(\zeta)$ vanishes in the end point k_0 and is Lipschitz continuous on $[k_0/2, k_0]$, because $R \in \mathcal{C}^1(\mathbb{R})$ and $|T(\zeta)|^2 = 1 - R(\zeta)\overline{R(\zeta)}$. Continuous extendability of $\psi(k)$ to k_0 thus follows from Theorem A.1. The integrability of the above functions is a straightforward consequence of, for example, L'Hôpital's rule. This yields the claimed representation of $\psi(k_0, k_0)$, since $\psi(k_0, k_0) = \lim_{k \searrow k_0} \psi(k, k_0)$ and dominated convergence applies.

□

3.3. Conjugation and Contour Deformation

After the preparations of the last section, we may now perform the conjugation and the contour deformation.
To this end, define the matrix $D(k)$ by

$$D(k) := \begin{cases} D_{\kappa_j,\check{\epsilon},-\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}(k) \begin{pmatrix} \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j} T(k,k_0)^{-1} & 0 \\ 0 & \frac{k-\mathrm{i}\kappa_j}{k+\mathrm{i}\kappa_j} T(k,k_0) \end{pmatrix}, & |k-\mathrm{i}\kappa_j| < \check{\epsilon}, \\ D_{\kappa_j,\check{\epsilon},-\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}}(k) \begin{pmatrix} \frac{k+\mathrm{i}\kappa_j}{k-\mathrm{i}\kappa_j} T(k,k_0)^{-1} & 0 \\ 0 & \frac{k-\mathrm{i}\kappa_j}{k+\mathrm{i}\kappa_j} T(k,k_0) \end{pmatrix}, & |k+\mathrm{i}\kappa_j| < \check{\epsilon}, \\ \begin{pmatrix} T(k,k_0)^{-1} & 0 \\ 0 & T(k,k_0) \end{pmatrix}, & \text{else.} \end{cases}$$

Conjugating $\check{m}(k)$ with $D(k)$ leads to

$$\tilde{m}(k) = \check{m}(k) D(k).$$

From the previous, i.e. Lemma 3.2, Theorem 3.3 and Lemma 3.4, we conclude that $\tilde{m}$ has the following properties:

(i) $\tilde{m}$ is holomorphic on $\mathbb{C}\backslash(\mathbb{R} \cup \bigcup_{j=1}^N \{|k-\mathrm{i}\kappa_j| = \check{\epsilon}\} \cup \{|k+\mathrm{i}\kappa_j| = \check{\epsilon}\})$.

(ii) $\tilde{m}$ can be extended continuously from the left resp. right to $\mathbb{R}\backslash\{-k_0, 0, k_0\}$ and to the circles around the points $\pm\mathrm{i}\kappa_j$. The extensions satisfy the jump condition $\tilde{m}_+ = \tilde{m}_- \tilde{v}$, where the jump matrix $\tilde{v}$ equals

$$\tilde{v}(k) = \begin{pmatrix} 1 & -\frac{k-\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}} T(k,k_0)^2 \\ 0 & 1 \end{pmatrix}, \qquad |k-\mathrm{i}\kappa_j| = \check{\epsilon}$$

$$\tilde{v}(k) = \begin{pmatrix} 1 & 0 \\ -\frac{k+\mathrm{i}\kappa_j}{\mathrm{i}\gamma_j^2 \mathrm{e}^{t\Phi(\mathrm{i}\kappa_j)}} T(k,k_0)^{-2} & 1 \end{pmatrix}, \qquad |k+\mathrm{i}\kappa_j| = \check{\epsilon}$$

on the circles and is given by

$$\tilde{v}(k) = \begin{pmatrix} 1 & -\overline{R(k)} \mathrm{e}^{-t\Phi(k)} T_-(k,k_0) T_+(k,k_0) \\ \frac{R(k)\mathrm{e}^{t\Phi(k)}}{T_-(k,k_0)T_+(k,k_0)} & 1-|R(k)|^2 \end{pmatrix}$$

for $-k_0 < k < 0$ or $0 < k < k_0$ and

$$\tilde{v}(k) = \begin{pmatrix} 1-|R(k)|^2 & -\overline{R(k)} \mathrm{e}^{-t\Phi(k)} T(k,k_0)^2 \\ R(k)\mathrm{e}^{t\Phi(k)} \frac{1}{T(k,k_0)^2} & 1 \end{pmatrix}$$

for $k < -k_0$ or $k > k_0$.

(iii) $\tilde{m}$ is bounded near the points $\pm k_0$ and near the origin.

(iv) The behavior at infinity is asymptotically given by

$$\tilde{m}(k) = (1,1) + O\left(\frac{1}{k}\right), \qquad \text{as } |k| \to \infty. \tag{3.9}$$

(v) $\tilde{m}$ is symmetric, i.e. it satisfies (2.13).

To prove for example the asymptotics at infinity, one can use (3.6). Symmetry is a direct consequence of the matrix symmetry condition

$$D(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} D(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.10}$$

The boundedness of $\tilde{m}$ near the origin in the upper half plane follows from (3.5) and the boundedness of $T(k)$. The factor $T(k)^{-1}$ may be unbounded near zero, but cancels using the definition of m. Symmetry then implies the boundedness also in the lower half plane.

As desired, the new jump matrix $\tilde{v}$ is converging exponentially to $\mathbb{I}$ on the small circles. In view of the observation presented in the last section, we aim for a decomposition of $\tilde{v}$ on $\mathbb{R}\backslash\{\pm k_0, 0\}$ to deal with the jump there. Now by our assumptions on $R(k)$, the two pairs of matrices

$$\tilde{b}^l(k) = \begin{pmatrix} 1 & 0 \\ R(k)\mathrm{e}^{t\Phi(k)}T(k,k_0)^{-2} & 1 \end{pmatrix}, \quad \tilde{b}^r(k) = \begin{pmatrix} 1 & R(-k)\mathrm{e}^{-t\Phi(k)}T(k,k_0)^2 \\ 0 & 1 \end{pmatrix},$$

and

$$\tilde{B}^l(k) = \begin{pmatrix} 1 & -\frac{T(k,k_0)^2 R(-k)\mathrm{e}^{-t\Phi(k)}}{1-R(k)R(-k)} \\ 0 & 1 \end{pmatrix}, \quad \tilde{B}^r(k) = \begin{pmatrix} 1 & 0 \\ -\frac{T(k,k_0)^{-2}R(k)\mathrm{e}^{t\Phi(k)}}{1-R(k)R(-k)} & 1 \end{pmatrix},$$

are holomorphic on the two strips $-\delta_R < \mathrm{Im}(k) < 0$ and $0 < \mathrm{Im}(k) < \delta_R$. An easy calculation yields that these matrices are continuously extendable from the left and right to $\mathbb{R}\backslash\{\pm k_0, 0\}$ and that $\tilde{v}$ can be factorized using the extensions. More precisely,

$$\tilde{v}(k) = \begin{cases} \tilde{b}^r_-(k)^{-1}\ \tilde{b}^l_+(k), & k \in \mathbb{R},\ |k| > k_0, \\ \tilde{B}^r_-(k)^{-1}\ \tilde{B}^l_+(k), & k \in \mathbb{R},\ 0 < |k| < k_0. \end{cases} \tag{3.11}$$

Thus a deformation step similar to the one sketched in Section 5 can be performed. In order to to so, fix a jump contour $\hat{\Sigma}$ consisting of $\hat{\Sigma}^1 = \hat{\Sigma}^1_l \cup \hat{\Sigma}^1_r$, $\hat{\Sigma}^2 = \hat{\Sigma}^2_l \cup \hat{\Sigma}^2_r$ and the small circles around the points $\mathrm{i}\kappa_j$ according to the following figure:

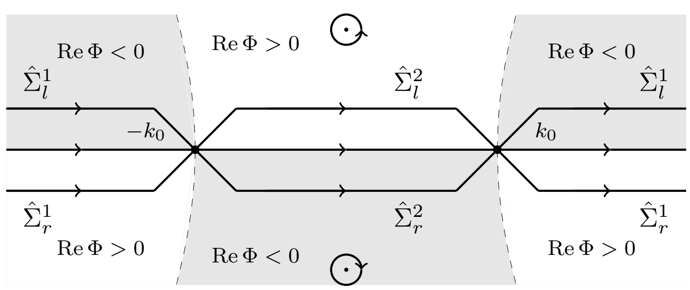


FIGURE 3.2. The contour $\hat{\Sigma}$.

As indicated in the figure, $\hat{\Sigma}^1 \cup \hat{\Sigma}^2$ should not intersect the circles around the points $\mathrm{i}\kappa_j$ and be fully contained in the strip $\{-\delta_R < \mathrm{Im}(k) < \delta_R\}$. For the last time in this chapter, we redefine $\tilde{m}$ and set

$$\hat{m}(k) := \begin{cases} \tilde{m}(k)\tilde{b}^l(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \hat{\Sigma}_l^1, \\ \tilde{m}(k)\tilde{b}^r(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \hat{\Sigma}_r^1, \\ \tilde{m}(k)\tilde{B}^l(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \hat{\Sigma}_l^2, \\ \tilde{m}(k)\tilde{B}^r(k)^{-1}, & k \text{ between } \mathbb{R} \text{ and } \hat{\Sigma}_r^2, \\ \tilde{m}(k), & k \in \mathbb{C}\backslash(\hat{\Sigma} \cup \mathbb{R}), \text{ else.} \end{cases}$$

Then the following properties are valid for $\hat{m}$:

(i) $\hat{m}$ can be extended to a holomorphic function on $\mathbb{C}\backslash\hat{\Sigma}$.

(ii) $\hat{m}$ can be extended continuously from the left resp. right to $\hat{\Sigma}\backslash\{\pm k_0\}$, and the extensions satisfy the jump condition $m_+ = m_-\hat{v}$ on $\hat{\Sigma}\backslash\{\pm k_0\}$, where

$$\hat{v}(k) = \begin{cases} \tilde{b}^l(k), & k \in \hat{\Sigma}_l^1\backslash\{\pm k_0\}, \\ \tilde{b}^r(k)^{-1}, & k \in \hat{\Sigma}_r^1\backslash\{\pm k_0\}, \\ \tilde{B}^l(k), & k \in \hat{\Sigma}_l^2\backslash\{\pm k_0\}, \\ \tilde{B}^r(k)^{-1}, & k \in \hat{\Sigma}_r^2\backslash\{\pm k_0\}, \\ \tilde{v}(k), & |k - \mathrm{i}\kappa_j| = \check{\epsilon} \text{ or } |k + \mathrm{i}\kappa_j| = \check{\epsilon}. \end{cases}$$

(iii) $\hat{m}$ is bounded near the points $\pm k_0$.

(iv) For every fixed $\epsilon > 0$, the behavior of $\hat{m}$ at infinity is given by

$$\hat{m}(k) = (1,1) + O\left(\frac{1}{k}\right), \qquad \text{as } |k| \to \infty,\ |\operatorname{Im}(k)| \geq \epsilon. \tag{3.12}$$

(v) $\hat{m}$ is symmetric, i.e. it satisfies (2.13).

In fact, the jump along $\mathbb{R}\backslash\{\pm k_0, 0\}$ vanishes and Morera's Theorem gives that $\hat{m}$ is holomorphic in $\mathbb{C}\backslash(\hat{\Sigma}\cup\{0\})$. Also, $1-R(k)R(-k)$ is analytic for $-\delta_R < \operatorname{Im}(k) < \delta_R$ and all other terms appearing in $\tilde{B}^l$ and $\tilde{m}$ are bounded, so $|\hat{m}(k)| \leq K|k|^n$ near the origin in the closed upper half plane. The same holds true for the lower half plane, if we use (3.7). Hence, the singularity at the origin cannot be essential. Again using (3.7), we find $T_\pm(k, k_0) = \overline{T_\mp(k, k_0)}^{-1}$ and therefore

$$\frac{T_+(k,k_0)^2}{1-|R(k)|^2} = T_+(k,k_0)T_-(k,k_0) = \frac{T_+(k,k_0)}{\overline{T_+(k,k_0)}}.$$

But this means $\hat{m}(k) = \tilde{m}_+(k)\tilde{B}^l_+(k)^{-1}$ is bounded on the negative real axis close to the origin, so the singularity must be removable.
The properties (ii) and (v) can be verified directly, whereas (iii) follows from Lemma 3.4. To deduce (iv), just notice that for $k = a + \imath b$, we have

$$t\operatorname{Re}(\Phi(k)) = 8tb^3 - 24ta^2b - 2xb. \tag{3.13}$$

Finally, we have completed the transformation. At first look, the resulting jump matrix problem is close to the identity for t large only in a very vague sense. The off-diagonal terms all contain an exponential term e^{ta}, where a has negative real part, but as the jump contour $\hat{\Sigma}$ depends on t as well, it makes no sense to ask whether $\hat{v}$ converges pointwise to the identity for $t \to \infty$. However, we can still meaningfully consider the L^p-norm $\|\hat{v}(x,t) - \mathbb{I}\|_{L^p(\hat{\Sigma}(x,t))}$. In Section 5.1, we will show that this norm indeed converges to zero for $t \to \infty$, if we consider values of (x, t) in a certain range and stay away from the real axis.

CHAPTER 4

Asymptotics for problems on a small cross

The goal of this chapter is to study Riemann–Hilbert problems posed on a small cross and derive an asymptotic expansion for their solutions. The rough idea will be to compare the Riemann Hilbert problem in question to an easier one. The solution of the simpler problem and the corresponding asymptotics can be explicitly constructed. Then we only have to transfer the asymptotics back to the original situation.

In the following, our jump contour will always be the infinite cross $\Sigma = \bigcup_{j=1}^{4} \Sigma_j$ consisting of the four line segments

$$\Sigma_1 = \{r\mathrm{e}^{-\mathrm{i}\pi/4};\ r \geq 0\}, \qquad \Sigma_2 = \{r\mathrm{e}^{\mathrm{i}\pi/4};\ r \geq 0\},$$
$$\Sigma_3 = \{r\mathrm{e}^{3\mathrm{i}\pi/4};\ r \geq 0\}, \qquad \Sigma_4 = \{r\mathrm{e}^{-3\mathrm{i}\pi/4};\ r \geq 0\}.$$

The segments Σ_j, $j = 1, ..., 4$, will be oriented as indicated by Figure 4.1. Now, let a holomorphic phase $\Theta\colon \mathbb{C} \to \mathbb{C}$, coefficient functions R_j on Σ_j, $j = 1, ..., 4$, and a parameter $\nu \geq 0$ be given. For $t > 0$, we define jump matrices

$$v_1 = \begin{pmatrix} 1 & -R_1(z)z^{2\mathrm{i}\nu}\mathrm{e}^{-t\Theta(z)} \\ 0 & 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 & 0 \\ R_2(z)z^{-2\mathrm{i}\nu}\mathrm{e}^{t\Theta(z)} & 1 \end{pmatrix},$$
$$v_3 = \begin{pmatrix} 1 & -R_3(z)z^{2\mathrm{i}\nu}\mathrm{e}^{-t\Theta(z)} \\ 0 & 1 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} 1 & 0 \\ R_4(z)z^{-2\mathrm{i}\nu}\mathrm{e}^{t\Theta(z)} & 1 \end{pmatrix},$$

where the principal branch of the logarithm on $\mathbb{C}\backslash\mathbb{R}_{\geq 0}$ with $-\pi < \arg(z) < \pi$ is used to define the power, and consider the related matrix Riemann–Hilbert problem

$$\begin{aligned} m_+(z) &= m_-(z)v_j(z), \qquad && z \in \Sigma_j, \quad j = 1, 2, 3, 4, \\ m(z) &\to \mathbb{I}, && z \to \infty. \end{aligned} \tag{4.1}$$

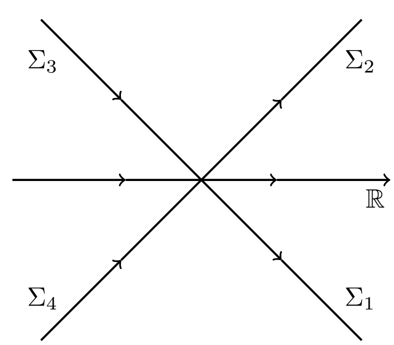


FIGURE 4.1. The contour Σ

More precisely, a function $m\colon \mathbb{C}\backslash\Sigma \to \mathbb{C}^{2\times 2}$ is said to solve the problem (4.1), if it satisfies the following four conditions:

(i) m is holomorphic on $\mathbb{C}\backslash\Sigma$.
(ii) m is bounded near the origin, i.e. m is bounded on $\{z \in \mathbb{C};\ |z| \le \epsilon\}\backslash\Sigma$ for some $\epsilon > 0$.
(iii) $m(z) \to \mathbb{I}$ uniformly for $z \in \mathbb{C}\backslash\Sigma$, $|z| \to \infty$.
(iv) For every $j = 1, ..., 4$, let Ω_+ resp. Ω_- denote the component of $\mathbb{C}\backslash\Sigma$ that lies next to Σ_j on the left resp. right side. Then m can be extended from Ω_+ to a continuous function m_+ on $\Omega_+ \cup (\Sigma_j\backslash\{0\})$ and from Ω_- to a continuous function m_- on $\Omega_- \cup (\Sigma_j\backslash\{0\})$ and the extensions satisfy

$$m_+(z) = m_-(z)v_j(z), \qquad z \in \Sigma_j\backslash\{0\}.$$

Throughout this chapter, whenever we use a notation analogous to (4.1) for a Riemann–Hilbert problem posed on Σ, we implicitly mean that a solution to the Riemann–Hilbert problem is defined via the above four conditions with the matrices v_j replaced by the corresponding jump matrices in condition (iv).

Note that in a way we are actually dealing with a family of Riemann–Hilbert problems indexed by $t > 0$. Imposing extra conditions on the phase and the coefficient functions leads to an asymptotic expansion for m in form of the next theorem.

Theorem 4.1. *Let a holomorphic phase Θ, a parameter $\nu \ge 0$ and coefficient functions R_j, $j = 1, ..., 4$, be given. Moreover, suppose there are constants ρ, L, L', C, $C' > 0$ and $r \in \mathbb{D}$ such that the following conditions are satisfied:*

(i) The coefficient R_j, $j = 1, ..., 4$, is continuous on Σ_j and compactly supported with

$$|R_j(z)| = 0 \qquad \text{for } z \in \Sigma_j,\ |z| \ge \rho.$$

The compatibility conditions

$$R_1(0) = \overline{r}, \qquad R_2(0) = r, \qquad R_3(0) = \frac{\overline{r}}{1-|r|^2}, \qquad R_4(0) = \frac{r}{1-|r|^2}, \tag{4.2}$$

and

$$\nu = -\frac{1}{2\pi}\log(1-|r|^2)$$

hold true. Furthermore, R_j, $j = 1, ..., 4$, is continuously differentiable on $\Sigma_j\backslash\{0\}$ and the derivative can be estimated by

$$|R_j'(z)| \le L + L'\left|\log(|z|)\right|. \tag{4.3}$$

(ii) The phase satisfies $\Theta(0) = \mathrm{i}\Theta_0 \in \mathrm{i}\mathbb{R}$ and within $|z| \le \rho$,

$$\pm\,\mathrm{Re}\left(\Theta(z)\right) \ge \frac{1}{4}|z|^2, \qquad \begin{cases} + & \text{for } z \in \Sigma_1 \cup \Sigma_3, \\ - & \text{else,} \end{cases} \tag{4.4}$$

$$|\Theta(z) - \Theta(0) - \frac{\mathrm{i}z^2}{2}| \le C|z|^3, \tag{4.5}$$

$$|\Theta'(z) - \mathrm{i}z| \le C'|z|^2. \tag{4.6}$$

Then there is some $T > 0$*, such that for* $t \geq T$ *the Riemann–Hilbert problem* (4.1) *is uniquely solvable and the solution* m *can be represented as*

$$m(z) = \mathbb{I} + \frac{1}{z}\frac{\mathrm{i}}{t^{1/2}}\begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix} + \frac{1}{z}e(z) + h(z), \tag{4.7}$$

where

$$\beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))}\mathrm{e}^{-\mathrm{i}t\Theta_0}t^{-\mathrm{i}\nu}$$

and the error terms $e(z)$ *and* $h(z)$ *have the following order:*

$$e(z) = O\left(t^{-\alpha}\right) \text{ for every } 1/2 < \alpha < 1 \quad \text{and} \quad h(z) = O\left(\frac{1}{tz^2}\right),$$

with estimates uniform with respect to $z \in \mathbb{C}\backslash\Sigma$ *and* t*.*

Suppose that $R_j(z)$*,* $j = 1, ..., 4$*,* $\Theta(z)$ *and* ν *depend on some parameter* λ *ranging over an index set* I*. Suppose further, that* $(\rho_\lambda)_\lambda$*,* $(L_\lambda)_\lambda$ $(L'_\lambda)_\lambda$*,* $(C_\lambda)_\lambda$*,* $(C'_\lambda)_\lambda$ *and* $(r_\lambda)_\lambda$ *are families of constants such that*

$$\sup_{\lambda \in I} L_\lambda + L'_\lambda + C_\lambda + C'_\lambda < \infty, \qquad \inf_{\lambda \in I} \rho_\lambda > 0, \qquad \sup_{\lambda \in I} |r_\lambda| < 1,$$

and such that for every fixed $\lambda \in I$*, the constants* ρ_λ*,* L_λ*,* L'_λ*,* C_λ*,* C'_λ *and* r_λ *satisfy (i) and (ii) for* $\Theta(z)$*,* ν *and* $R_j(z)$*,* $j = 1, ..., 4$*, corresponding to* λ*.*
Then there is some $T > 0$*, such that for every* $t \geq T$ *and* $\lambda \in I$ *the Riemann–Hilbert problem* (4.1) *is uniquely solvable and the solution can be represented as in* (4.7) *with error terms* $e(z)$ *and* $h(z)$ *of the following order:*

$$e(z) = O\left(\, (L_\lambda + L'_\lambda + |r_\lambda|)\, t^{-\alpha}\right) \text{ for every } 1/2 < \alpha < 1 \quad \text{and} \quad h(z) = O\left(\frac{1}{tz^2}\right),$$

with estimates uniform with respect to $z \in \mathbb{C}\backslash\Sigma$*,* $t \geq T$ *and* $\lambda \in I$*.*

Clearly, the first statement of the theorem is a trivial consequence of the second one, so we will focus on the proof of the latter only. In this case, the phase, parameter and coefficient functions should actually carry the subindex λ as well as the constants, but as it was already done in the presentation of the theorem, we will suppress this index in order to shorten the notation.

Since $v_j(z) = \mathbb{I}$ for $|z| > \rho_\lambda$, the Riemann–Hilbert problem (4.1) is equivalent to a Rieman-Hilbert on the small cross $\Sigma \cap \{z \in \mathbb{C};\ |z| \leq \rho_\lambda\}$, which motivates the title of this chapter.

Now, checking that $\det(v_j) \equiv 1$ we see that uniqueness of solutions follows immediately by the usual Liouville argument (see the proof of Theorem 4.6). The proof of the remaining statement will be given in the rest of this chapter.

4.1. The connection to singular integral equations on H^1 spaces

In this section, we use the Cauchy operator $\mathcal{C}$ to transform Riemann–Hilbert problems into singular integral equations on H^1 spaces.

First of all, we notice that after reversing the orientation on the two line segments Σ_1 and Σ_4, the set $\Sigma \cup \{\infty\}$ is a Carleson jump contour as in Appendix A. It follows that the statements in Theorem A.4 also hold true for $\Gamma = \Sigma$ and the

Cauchy operator defined with the original orientation on Σ. However, these results concerning the space $L^2(\Sigma)$ will not be sufficient in our later argument. We need to study the properties of the Cauchy operator on the respective H^1 space instead.

We denote by $H^1(\Sigma_j\backslash\{0\})$, $j = 1, ..., 4$, the space of all functions f on $\Sigma_j\backslash\{0\}$ such that f and its distributional derivative f' belong to $L^2(\Sigma_j\backslash\{0\})$. It follows from the Sobolev imbedding theorem (cf. e.g. [**2**]) that any element of $H^1(\Sigma_j\backslash\{0\})$ has a continuous representative that is continuously extendable to the whole of Σ_j and tends to zero as $|z|$ tends to infinity. Several Sobolev inequalities apply in this situation, an important one in the context here is

$$\|f\|_{L^\infty(\Sigma_j\backslash\{0\})} \leq K\left(\|f\|^2_{L^2(\Sigma_j\backslash\{0\})} + \|f'\|^2_{L^2(\Sigma_j\backslash\{0\})}\right)^{1/2}, \quad f \in H^1(\Sigma_j\backslash\{0\}). \tag{4.8}$$

Similarly, we write $H^1(\Sigma\backslash\{0\})$ for the space of all functions f on $\Sigma\backslash\{0\}$ such that for every $j = 1, ..., 4$ the restriction f_j of f to $\Sigma_j\backslash\{0\}$ is in $H^1(\Sigma_j\backslash\{0\})$. For $f \in H^1(\Sigma\backslash\{0\})$ we introduce the norm

$$\|f\|_{H^1(\Sigma\backslash\{0\})} := \sum_{j=1}^{4}\left(\|f_j\|^2_{L^2(\Sigma_j\backslash\{0\})} + \|f_j'\|^2_{L^2(\Sigma_j\backslash\{0\})}\right)^{1/2}.$$

Using this new terminology, we have the following lemma (see [**4**], pp. 87–90):

Lemma 4.2. *Suppose $f \in H^1(\Sigma\backslash\{0\})$ satisfies*

$$\sum_{j=1}^{4}\sigma_j \lim_{z\to 0, z\in\Sigma_j} f_j(z) = 0, \tag{4.9}$$

where $\sigma_j = 1$, if Σ_j is oriented pointing away from 0 and $\sigma_j = -1$, if Σ_j is oriented pointing towards 0. Then

$$\frac{d}{dz}(\mathcal{C}f)(z) = (\mathcal{C}f')(z), \qquad z \in \mathbb{C}\backslash\Sigma, \tag{4.10}$$

and $\mathcal{C}f$ is uniformly bounded by

$$|(\mathcal{C}f)(z)| \;\leq\; \|f\|_{H^1(\Sigma\backslash\{0\})}, \qquad z \in \mathbb{C}\backslash\Sigma. \tag{4.11}$$

Moreover, $(\mathcal{C}f)(z)$ tends to zero uniformly for $|z| \to \infty$ and

$$|(\mathcal{C}f)(x) - (\mathcal{C}f)(y)| \leq |x-y|^{1/2}\,\sqrt{2}\,\sum_{j=1}^{4}\|f_j'\|_{L^2(\Sigma_j\backslash\{0\})} \tag{4.12}$$

if x, y are in the same component of $\mathbb{C}\backslash\Sigma$, so f is uniformly Hölder continuous of order $1/2$ on every component of $\mathbb{C}\backslash\Sigma$.

Proof. For the proof of (4.10), a straightforward computation gives

$$\begin{aligned}
\frac{d}{dz}\,(Cf)(z) &= \frac{d}{dz}\left(\frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{4}\sigma_j\int_0^\infty \frac{f(rz_j)}{rz_j - z}\,z_j\,dr\right)\\
&= \frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{4}\sigma_j\int_0^\infty \frac{f(rz_j)}{(rz_j-z)^2}\,z_j\,dr\\
&= \frac{1}{2\pi \mathrm{i}}\sum_{j=1}^{4}\left(\sigma_j\lim_{r\to 0}\frac{f(rz_j)}{z - rz_j} + \sigma_j\int_0^\infty \frac{f'(rz_j)}{rz_j - z}\,z_j\,dr\right)
\end{aligned}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{\Sigma\setminus\{0\}} \frac{f'(s)}{s-z}\, ds + \frac{1}{2\pi \mathrm{i} z} \sum_{j=1}^{4} \left(\sigma_j \lim_{z\to 0, z\in\Sigma_j\setminus\{0\}} f(z) \right) = (Cf')(z).$$

To derive the pointwise estimate (4.11), we start by introducing an operator A_z on $L^2((0,\infty))$ similar to the Cauchy operator and prove that A_z is a bounded operator from $L^2((0,\infty))$ to $L^2((0,\infty))$. The key idea here will be to observe that under a suitable transform the operator A_z is equivalent to a multiplication operator with a bounded function. So, for $z = \mathrm{e}^{\mathrm{i}\theta}, 0 < \theta < 2\pi$ and $f \in L^2((0,\infty))$ let

$$(A_z f)(x) := \int_0^\infty \frac{f(r)}{x - zr}\, dr, \quad x > 0.$$

Now for $f \in L^2((0,\infty))$, define the Mellin transform $\mathcal{M}f$ by

$$(\mathcal{M}f)\ (s) = \mathcal{F}(\mathrm{e}^{t/2} f(\mathrm{e}^t))\ (-s) = \mathcal{F}^3(\mathrm{e}^{t/2} f(\mathrm{e}^t))\ (s), \quad s \in \mathbb{R}.$$

Here, $\mathcal{F}$ denotes the Fourier transform on $L^2((-\infty,\infty))$ with norming constants such that

$$\mathcal{F}(g)\ (s) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}st} g(t) \frac{dt}{\sqrt{2\pi}}, \qquad g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),\ s \in \mathbb{R}.$$

It should be remarked, that if $f \in L^2((0,\infty))$ is such that $\mathrm{e}^{t/2} f(\mathrm{e}^t)$ is in $L^1((-\infty,\infty))$ (or equivalently, $x^{-1/2} f(x)$ is in $L^1((0,\infty))$), then $\mathcal{M}f$ is given by the more common formula

$$(\mathcal{M}f)\ (s) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}st + t/2} f(\mathrm{e}^t) \frac{dt}{\sqrt{2\pi}} = \int_0^\infty x^{-1/2+\mathrm{i}s} f(x) \frac{dx}{\sqrt{2\pi}}.$$

Since by the substitution $x = \mathrm{e}^t$ the mapping $f(x) \mapsto \mathrm{e}^{t/2} f(\mathrm{e}^t)$ is a unitary transformation between $L^2((0,\infty))$ and $L^2((-\infty,\infty))$, $\mathcal{M}$ is a unitary transform between $L^2((0,\infty))$ and $L^2((-\infty,\infty))$. The inverse is then given by

$$(\mathcal{M}^{-1} g)\ (x) = x^{-1/2}\ (\mathcal{F}g)(\log x),$$

which by the continuity of $\mathcal{M}^{-1}$ equals the $L^2((0,\infty))$ limit of the sequence

$$\mathcal{M}^{-1}(g\chi_{(-n,n)})\ (x) = \int_{-n}^{n} x^{-1/2-\mathrm{i}s} g(s) \frac{ds}{\sqrt{2\pi}}.$$

Now fix $x > 0$. The continuity of the inner product on $L^2((0,\infty))$ and Fubini's theorem lead to

$$(A_z \mathcal{M}^{-1} g)\ (x) = \int_0^\infty \frac{\mathcal{M}^{-1} g\ (r)}{x - zr}\, dr = \lim_{n\to\infty} \int_0^\infty \int_{-n}^{n} \frac{r^{-1/2-\mathrm{i}s}}{x - zr} g(s)\ \frac{ds}{\sqrt{2\pi}}\ dr$$
$$= \lim_{n\to\infty} \int_{-n}^{n} g(s) \int_0^\infty \frac{r^{-1/2-\mathrm{i}s}}{x - zr}\ dr\ \frac{ds}{\sqrt{2\pi}}.$$

By a residue computation that we postpone until the end of the proof,

$$\int_0^\infty \frac{r^{-1/2-\mathrm{i}s}}{x - zr}\ dr = \frac{2\pi \mathrm{i}}{1 + \mathrm{e}^{-2\pi s}}\ z^{-1/2+\mathrm{i}s} x^{-1/2-\mathrm{i}s}, \tag{4.13}$$

where we use the usual branch of the logarithm on $\mathbb{C}\setminus\mathbb{R}_{\geq 0}$ to define the power, namely we set $\log(z) := \log(|z|) + \mathrm{i}\arg(z)$ with $0 < \arg(z) < 2\pi$. Then

$$(A_z \mathcal{M}^{-1} g)\ (x) = \lim_{n\to\infty} \int_{-n}^{n} x^{-1/2-\mathrm{i}s} \frac{2\pi \mathrm{i}}{1 + \mathrm{e}^{-2\pi s}} z^{-1/2+\mathrm{i}s} g(s) \frac{ds}{\sqrt{2\pi}}$$

$$= \lim_{n\to\infty} (\mathcal{M}^{-1} M_z(g\chi_{(-n,n)}))(x),$$

where M_z denotes the multiplication operator on $L^2((-\infty,\infty))$ corresponding to the bounded function $m_z(s) = \frac{2\pi\mathrm{i}}{1+\mathrm{e}^{-2\pi s}} z^{-1/2+\mathrm{i}s} = \frac{2\pi\mathrm{i}}{1+\mathrm{e}^{-2\pi s}} \mathrm{e}^{-\mathrm{i}\theta/2}\mathrm{e}^{-\theta s}$.
Since $x > 0$ was arbitrary, the sequence $\mathcal{M}^{-1}M_z\ (g\chi_{(-n,n)})$ converges pointwise almost everywhere to $A_z\mathcal{M}^{-1}g$, but also tends to $\mathcal{M}^{-1}M_z g$ in $L^2((0,\infty))$, which yields $A_z\mathcal{M}^{-1}g = \mathcal{M}^{-1}M_z g$ if we reduce to an almost everywhere convergent subsequence. Hence, A_z is well-defined as an operator from $L^2((0,\infty))$ to $L^2((0,\infty))$ and allows the estimate

$$\|A_z\| = \|M_z\| \ \le\ \sup_{s\in\mathbb{R}} |m_z(s)| \ \le\ 2\pi.$$

In our next step, we will estimate the L^2 norm of the Cauchy operator along any ray not contained in the cross Σ. To this end, let $f \in L^2(\Sigma\backslash\{0\})$ and $z = \mathrm{e}^{\mathrm{i}\theta}, z \notin \Sigma$ and write z_j for the uniquely determined $z_j \in \Sigma_j$ satisfying $|z_j| = 1$. Setting $\Sigma_\theta = \{\lambda z;\ \lambda > 0\}$, we have

$$\begin{aligned}
\|Cf\|_{L^2(\Sigma_\theta)} &= \left(\int_0^\infty |Cf(tz)|^2 dt\right)^{1/2} \\
&\le \sum_{j=1}^4 \left(\int_0^\infty \left|\int_{\Sigma_j} \frac{f(\zeta)}{\zeta - tz}\frac{d\zeta}{2\pi\mathrm{i}}\right|^2 dt\right)^{1/2} \\
&= \sum_{j=1}^4 \left(\int_0^\infty \left|\int_0^\infty \frac{f(rz_j)}{t-(z_j/z)r}\ \frac{dr}{2\pi}\right|^2 dt\right)^{1/2} \\
&= \frac{1}{2\pi}\sum_{j=1}^4 \left(\int_0^\infty \left|A_{\frac{z_j}{z}} f(\cdot z_j)(t)\right|^2 dt\right)^{1/2} \\
&= \frac{1}{2\pi}\sum_{j=1}^4 \|A_{\frac{z_j}{z}} f(\cdot z_j)\|_{L^2((0,\infty))} \ \le\ \sum_{j=1}^4 \|f(\cdot z_j)\|_{L^2((0,\infty))} \\
&= \sum_{j=1}^4 \|f_j\|_{L^2(\Sigma_j\backslash\{0\})}.
\end{aligned} \tag{4.14}$$

Now, if f is also in $H^1(\Sigma\backslash\{0\})$ and satisfies (4.9), we can apply (4.10) to get a similar estimate for the derivative, namely $\left\|\frac{d}{dz}Cf\right\|_{L^2(\Sigma_\theta\backslash\{0\})} \le \sum_{j=1}^4 \|f_j'\|_{L^2(\Sigma_j\backslash\{0\})}$.
Finally, we can combine these results to get the desired pointwise estimate. For $z = e^{i\theta} \notin \Sigma$ and $t > 0$ the above yields

$$\begin{aligned}
|Cf(tz)|^2 &= \left|-\int_t^\infty (Cf(\cdot z)^2)'(r)\ dr\right| = \left|-2\int_t^\infty (Cf)(rz)\left(\frac{d}{dz}Cf\right)(rz)z\ dr\right| \\
&\le 2\|Cf\|_{L^2(\Sigma_\theta\backslash\{0\})}\ \|\frac{d}{dz}Cf\|_{L^2(\Sigma_\theta\backslash\{0\})} \le \|f\|^2_{H^1(\Sigma\backslash\{0\}}.
\end{aligned}$$

So the pointwise estimate is proven except for formula (4.13). To calculate the corresponding integral we first substitute $r = 1/t$ and get

$$\int_0^\infty \frac{r^{-1/2-\mathrm{i}s}}{x - zr}\ dr = \frac{1}{x}\int_0^\infty \frac{t^{-1/2+\mathrm{i}s}}{t - z/x}\ dt.$$

Next, set $a := z/x \in \mathbb{C}\backslash\mathbb{R}_{\geq 0}$ and consider the function $f(\zeta) := \frac{\zeta^{-1/2+\mathrm{i}s}}{\zeta - a}$ on $\mathbb{C}\backslash\mathbb{R}_{\geq 0}$, where as above the usual branch of the logarithm on $\mathbb{C}\backslash\mathbb{R}_{\geq 0}$ is used to define the power. We integrate f over the "pacman contour" γ_N given by Figure 4.2 (see page 28). For N large enough, Cauchy's theorem implies $\int_{\gamma_N} f(\zeta)\, d\zeta = 2\pi\mathrm{i}a^{-1/2+\mathrm{i}s}$. Now letting N tend to infinity,

$$\lim_{N\to\infty} \int_{\gamma_1} f(\zeta)\, d\zeta = \lim_{N\to\infty} \int_{1/N}^{N} f(t\mathrm{e}^{\mathrm{i}/N})\mathrm{e}^{\mathrm{i}/N}\, dt$$
$$= \lim_{N\to\infty} \mathrm{e}^{1/N(\mathrm{i}/2-s)} \int_{1/N}^{N} \frac{t^{-1/2+\mathrm{i}s}}{t\mathrm{e}^{\mathrm{i}/N} - a}\, dt \quad = \int_0^\infty \frac{t^{-1/2+\mathrm{i}s}}{t-a}\, dt,$$

where dominated convergence can be applied in order to justify the last step. Similarly,

$$\lim_{N\to\infty} \int_{\gamma_3} f(\zeta)\, d\zeta = \mathrm{e}^{-2\pi s} \int_0^\infty \frac{t^{-1/2+\mathrm{i}s}}{t-a}\, dt.$$

Note that the integrals over γ_2 and γ_3 tend to zero as N tends to ∞. All in all we have proven $2\pi\mathrm{i}a^{-1/2+\mathrm{i}s} = (1+\mathrm{e}^{-2\pi s}) \int_0^\infty \frac{t^{-1/2+\mathrm{i}s}}{t-a}\, dt$, which together with the above substitution implies (4.13).

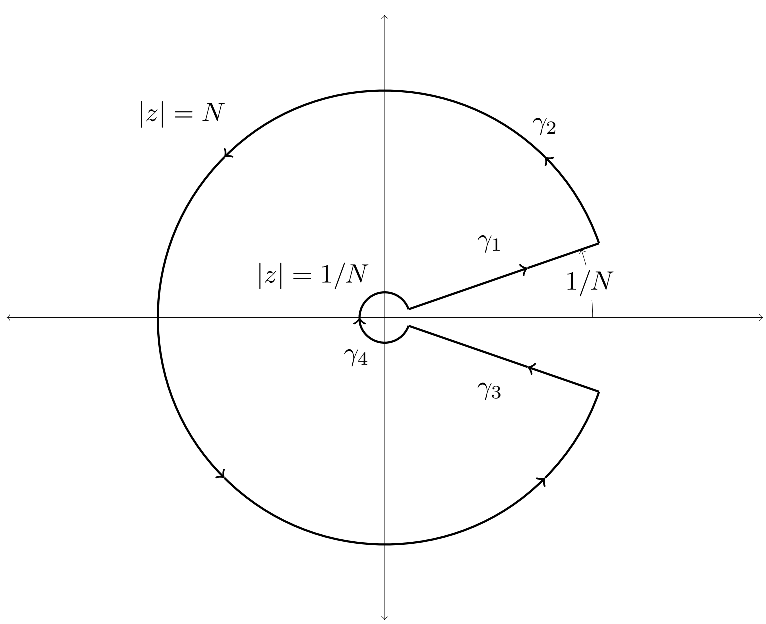


FIGURE 4.2. The contour γ_N

To see that $\mathcal{C}f$ tends to zero uniformly, choose $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ such that $\chi(s) = 1$ for $|s| \leq 1$, $\chi(s) = 0$ for $|s| \geq 2$ and set $\chi_r(s) = \chi(s/r)$ whenever $r > 0$, $s \in \mathbb{R}^2$. Now let $\epsilon > 0$ be given. Since $\|(1-\chi_r)f\|_{H^1(\Sigma\backslash\{0\})}$ tends to zero for $r \to \infty$, there is some $R > 0$ such that $\|(1-\chi_R)f\|_{H^1(\Sigma\backslash\{0\})} \leq \epsilon/2$. Assuming

$|z| \geq \max\{4R,\ (2\,\|\chi_R\|_{L^2(\Sigma)}\,\|f\|_{L^2(\Sigma)})\,/\,(\epsilon\pi)\}$, we have

$$\begin{aligned}|(\mathcal{C}f)(z)| \;&\leq\; \left|\frac{1}{2\pi\mathrm{i}}\int_\Sigma \chi_R f\ \frac{ds}{s-z}\right| + \left|\frac{1}{2\pi\mathrm{i}}\int_\Sigma (1-\chi_R) f\ \frac{ds}{s-z}\right| \\ &\leq\; \frac{1}{|z|}\,\frac{1}{\pi}\int_\Sigma |\chi_R f|\,|ds| + \|\chi_R f\|_{H^1(\Sigma\setminus\{0\})} \\ &\leq\; \frac{1}{|z|}\,\frac{1}{\pi}\|\chi_R\|_{L^2(\Sigma)}\|f\|_{L^2(\Sigma)} + \frac{\epsilon}{2} \leq \epsilon,\end{aligned}$$

where (4.11) and the estimate $|z-s| \geq |z|/2$ for s within the support of χ_R have been used.

Finally, we turn to the proof of the Hölder continuity of $\mathcal{C}f$. To this end, suppose x, y are in the same component of $\mathbb{C}\backslash\Sigma$. Because $\mathcal{C}f$ is continuous, it may be assumed that $x \neq y$ and that the line segment from x to y is not parallel to any of the segments Σ_j, $j = 1, ..., 4$. Remembering (4.10),

$$\begin{aligned}|(\mathcal{C}f)(x) - (\mathcal{C}f)(y)| &= \left|\int_x^y (\mathcal{C}f')(z)\ dz\right| \\ &\leq |x-y|^{1/2}\ \left(\int_x^y |(\mathcal{C}f')(z)|^2\ |dz|\right)^{1/2} \\ &\leq |x-y|^{1/2}\ \sum_{j=1}^{4}\left(\int_x^y \left|\frac{1}{2\pi\mathrm{i}}\int_{\Sigma_j}\frac{f_j'(s)}{s-z}\ ds\right|^2\ |dz|\right)^{1/2},\end{aligned}$$

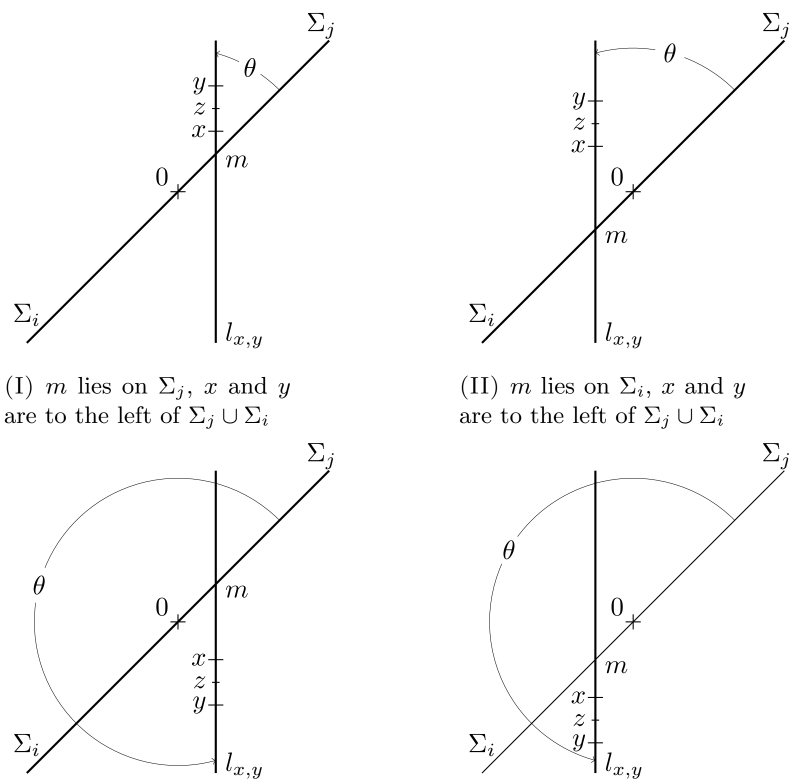
which means we only need to show

$$\left(\int_x^y \left|\frac{1}{2\pi\mathrm{i}}\int_{\Sigma_j}\frac{f_j'(s)}{s-z}\ ds\right|^2\ |dz|\right)^{1/2} \leq \sqrt{2}\ \|f_j'\|_{L^2(\Sigma_j\setminus\{0\})}$$

for $j = 1, ..., 4$. To do so, fix j and recall z_j was defined to be the uniquely determined element $z_j \in \Sigma_j$ with $|z_j| = 1$. Denote the segment of Σ that is opposite to Σ_j by Σ_i. We orient $\Sigma_j \cup \Sigma_i$ pointing from the infinite point on Σ_i to the infinite point on Σ_j, but keep the original orientation on the segments Σ_j and Σ_i. If we expand the line segment from x to y to a straight line $l_{x,y}$, then $l_{x,y}$ intersects $\Sigma_j \cup \Sigma_i$ in a unique point $m \in \mathbb{C}$. Depending on whether m lies on Σ_i or Σ_j and on whether i and j are to the left or right side of $\Sigma_j \cup \Sigma_i$, we have one of four possible configurations (cf. Figure 4.3 on page 30). We use θ to denote the angle defined by this figure. Now, let

$$d = \begin{cases} |m|, & m \in \Sigma_j, \\ -|m|, & m \in \Sigma_i, \end{cases}$$

to get $m = d\, z_j$, expand f_j' to $\Sigma_j \cup \Sigma_i$ by $f_j' \equiv 0$ on Σ_i and put $g^\pm(s) := f_j'(m \pm s z_j)$ for $s > 0$ to obtain two functions g^+ and g^- in $L^2((0,\infty))$.



(I) m lies on Σ_j, x and y are to the left of $\Sigma_j \cup \Sigma_i$

(II) m lies on Σ_i, x and y are to the left of $\Sigma_j \cup \Sigma_i$

(III) m lies on Σ_j, x and y are to the right of $\Sigma_j \cup \Sigma_i$

(IV) m lies on Σ_i, x and y are to the right of $\Sigma_j \cup \Sigma_i$

FIGURE 4.3. The four possible configurations

For z on the line segment from x to y we write $z = m + |z-m|\mathrm{e}^{\mathrm{i}\theta} z_j$ and substitute $t = d + r$ to calculate

$$
\begin{aligned}
\sigma_j \int_{\Sigma_j} \frac{f_j'(s)}{s-z}\, ds &= \int_{-\infty}^{d} \frac{f_j'(tz_j)}{tz_j - z} z_j\, dt + \int_d^{\infty} \frac{f_j'(tz_j)}{tz_j - z} z_j\, dt \\
&= \int_{-\infty}^{0} \frac{f_j'(m + rz_j)}{m + rz_j - z} z_j\, dr + \int_0^{\infty} \frac{f_j'(m + rz_j)}{m + rz_j - z} z_j\, dr \\
&= \int_0^{\infty} \frac{f_j'(m - rz_j)}{m - rz_j - z} z_j\, dr + \int_0^{\infty} \frac{f_j'(m + rz_j)}{m + rz_j - z} z_j\, dr \\
&= \int_0^{\infty} \frac{g^-(r)}{-|z-m|\mathrm{e}^{\mathrm{i}\theta} - r}\, dr + \int_0^{\infty} \frac{g^+(r)}{-|z-m|\mathrm{e}^{\mathrm{i}\theta} + r}\, dr \\
&= -\frac{1}{\mathrm{e}^{\mathrm{i}\theta}} \left[(A_{\mathrm{e}^{-\mathrm{i}\theta}} g^+)(|z-m|) + (A_{\mathrm{e}^{-\mathrm{i}\theta - \mathrm{i}\pi}} g^-)(|z-m|) \right].
\end{aligned}
$$

Noting that $\sqrt{a}+\sqrt{b} \le \sqrt{2}\sqrt{a+b}$ for $a, b \ge 0$, the desired estimate can be obtained by

$$\begin{aligned}
&\left(\int_x^y \left| \frac{1}{2\pi\mathrm{i}} \int_{\Sigma_j} \frac{f_j'(s)}{s-z}\, ds \right|^2 \, |dz| \right)^{1/2} \\
&= \frac{1}{2\pi} \left(\int_x^y \left| (A_{\mathrm{e}^{-\mathrm{i}\theta}} g^+)(|z-m|) + (A_{\mathrm{e}^{-\mathrm{i}\theta-\mathrm{i}\pi}} g^-)(|z-m|) \right|^2 \, |dz| \right)^{1/2} \\
&\le \frac{1}{2\pi} \left(\int_0^\infty \left| (A_{\mathrm{e}^{-\mathrm{i}\theta}} g^+)(t) + (A_{\mathrm{e}^{-\mathrm{i}\theta-\mathrm{i}\pi}} g^-)(t) \right|^2 \, dt \right)^{1/2} \\
&\le \|g^+\|_{L^2((0,\infty))} + \|g^-\|_{L^2((0,\infty))} \\
&\le \sqrt{2}\, (\|g^+\|^2_{L^2((0,\infty))} + \|g^-\|^2_{L^2((0,\infty))})^{1/2} = \sqrt{2}\, \|f_j'\|_{L^2(\Sigma_j \setminus \{0\})},
\end{aligned}$$

where we also used the parametrization $z = m + t\mathrm{e}^{\mathrm{i}\theta} z_j$ combined with the previous estimate for the operator norms of $A_{\mathrm{e}^{-\mathrm{i}\theta}}$ and $A_{\mathrm{e}^{-\mathrm{i}\theta-\mathrm{i}\pi}}$. □

This permits us to establish a connection between Riemann–Hilbert problems and singular integral equations. For $f\colon \Sigma \to \mathbb{C}^{2\times 2}$, we will write $f \in L^2(\Sigma)$ whenever all components of f are in $L^2(\Sigma)$ and set

$$\|f\|_2 = \max_{i,j=1,2} \|f_{ij}\|_2.$$

In similar situations, we use the analogous notation. When f is matrix-valued, $f \in L^2(\Sigma)$, we define the Cauchy operator $\mathcal{C}f$ componentwise and obtain a 2×2 matrix valued function on $\mathbb{C}\setminus\Sigma$ with analytical entries.

Lemma 4.3. *Assume a jump matrix v on Σ with $\det(v) \equiv 1$ such that $w = v - \mathbb{I}$ is continuous on Σ, continuously differentiable on $\Sigma_j\setminus\{0\}$ and satisfies $w(0) = 0$ and $w \in L^\infty(\Sigma) \cap L^2(\Sigma\setminus\{0\})$, $w' \in L^2(\Sigma\setminus\{0\})$. Then $\mathcal{C}_- w \in H^1(\Sigma\setminus\{0\})$ and*

$$\begin{aligned}
\mathcal{C}_w\colon H^1(\Sigma\setminus\{0\}) &\longrightarrow H^1(\Sigma\setminus\{0\}) \\
f &\mapsto \mathcal{C}_-(fw)
\end{aligned}$$

is a well-defined, bounded operator. Also, there are constants $c, c' > 0$ independent of v such that

$$\|\mathcal{C}_w(f)\|_{H^1(\Sigma\setminus\{0\})} \le c \max\{\|w\|_\infty, \|w'\|_{L^2(\Sigma\setminus\{0\})}\}\, \|f\|_{H^1(\Sigma\setminus\{0\})}, \quad f \in H^1(\Sigma\setminus\{0\}),$$

and

$$\|\mathcal{C}_- w\|_{H^1(\Sigma\setminus\{0\})} \le c' \|w\|_{H^1(\Sigma\setminus\{0\})}.$$

Assume, in addition, that $\mu - \mathbb{I} \in H^1(\Sigma\setminus\{0\})$ solves the singular integral equation

$$(\mathrm{id} - \mathcal{C}_w)(\mu - \mathbb{I}) = \mathcal{C}_- w \qquad \text{in } H^1(\Sigma\setminus\{0\}). \tag{4.15}$$

Then the unique solution m to the Riemann Hilbert problem with jump matrix v is given by

$$m(z) = \mathbb{I} + \frac{1}{2\pi\mathrm{i}} \int_\Sigma \mu(s) w(s) \frac{ds}{s-z}, \qquad z \in \mathbb{C}\setminus\Sigma, \tag{4.16}$$

and

$$m_- = \mu.$$

PROOF. First of all, we prove that $\mathcal{C}_- w \in H^1(\Sigma\backslash\{0\})$ and that $\mathcal{C}_w$ is well-defined. It is sufficient to observe that for any $g \in H^1(\Sigma\backslash\{0\})$ which satisfies (4.9) we have $\mathcal{C}_- g \in H^1(\Sigma\backslash\{0\})$ and $(\mathcal{C}_- g)' = \mathcal{C}_-(g')$. So, let for instance $\phi \in C_0^\infty(\Sigma_1\backslash\{0\})$ be given. Then there is some $N \in \mathbb{N}$ such that $\phi(z) = 0$ whenever $z \in \Sigma_1$ and $|z| \leq 1/N$ or $|z| \geq N$. Hence by partial integration and (4.10)

$$\begin{aligned} -\int_{\Sigma_1} (\mathcal{C}_- g)(z) \frac{d\phi}{dz}(z)\, dz &= -\int_{1/N}^{N} (\mathcal{C}_- g)(r\mathrm{e}^{-\mathrm{i}\pi/4}) \frac{d\phi}{dz}(r\mathrm{e}^{-\mathrm{i}\pi/4}) \mathrm{e}^{-\mathrm{i}\pi/4}\, dr \\ &= \lim_{t\to 0} -\int_{1/N}^{N} (\mathcal{C} g)(t\mathrm{e}^{-3\mathrm{i}\pi/4} + r\mathrm{e}^{-\mathrm{i}\pi/4}) \frac{d\phi}{dz}(r\mathrm{e}^{-\mathrm{i}\pi/4}) \mathrm{e}^{-\mathrm{i}\pi/4}\, dr \\ &= \lim_{t\to 0} \int_{1/N}^{N} (\mathcal{C} g')(t\mathrm{e}^{-3\mathrm{i}\pi/4} + r\mathrm{e}^{-\mathrm{i}\pi/4}) \phi(r\mathrm{e}^{-\mathrm{i}\pi/4}) \mathrm{e}^{-\mathrm{i}\pi/4}\, dr \\ &= \int_{1/N}^{N} (\mathcal{C}_- g')(r\mathrm{e}^{-\mathrm{i}\pi/4}) \phi(r\mathrm{e}^{-\mathrm{i}\pi/4}) \mathrm{e}^{-\mathrm{i}\pi/4}\, dr \\ &= \int_{\Sigma_1} (\mathcal{C}_- g')(z)\phi(z)\, dz. \end{aligned}$$

In fact, passing to the limit for $t \to 0$ is allowed here, because the Cauchy operator $\mathcal{C}g$ resp. $\mathcal{C}g'$ consists of the integral over $\Sigma_1 \cup \Sigma_3$, which converges in L^2 by Theorem A.2, and the integral over $\Sigma_2 \cup \Sigma_4$, where dominated convergence can be applied. If we repeat this argument for the other segments of Σ, we obtain the desired conclusion.

The boundedness of $\mathcal{C}_w$ and the second statement are then easily obtained via the Sobolev inequality (4.8) and the continuity of $\mathcal{C}_- : L^2(\Sigma) \to L^2(\Sigma)$.

For the final part, suppose $(\mu - \mathbb{I}) \in H^1(\Sigma\backslash\{0\})$ solves the singular integral equation (4.16). Then define m as above, i.e. $m = \mathbb{I} + \mathcal{C}((\mu - \mathbb{I})w) + \mathcal{C}(w)$. In view of Lemma 4.2, the conditions (i), (ii) and (iii) of the Riemann–Hilbert and the existence of the continuous limits m_+ and m_- clearly hold true. Calculating the continuous limit taken from the right side to $\Sigma\backslash\{0\}$,

$$\begin{aligned} m_-(z) &= \mathbb{I} + \mathcal{C}_-((\mu - \mathbb{I})w)\ (z) + \mathcal{C}_-(w)\ (z) \\ &= \mathbb{I} + \mathcal{C}_-((\mu - \mathbb{I})w)\ (z) + (\mathrm{id} - \mathcal{C}_w)(\mu - \mathbb{I})\ (z) = \mu(z). \end{aligned}$$

almost everywhere on $\Sigma\backslash\{0\}$. But both sides of the equation are continuous, so $m_- = \mu$ on $\Sigma\backslash\{0\}$ as claimed above. Since $\mathrm{id} = \mathcal{C}_+ - \mathcal{C}_-$ on $L^2(\Sigma)$ by Theorem A.4,

$$\begin{aligned} m_+(z) &= \mathbb{I} + \mathcal{C}_+((\mu - \mathbb{I})w)\ (z) + \mathcal{C}_+(w)\ (z) \\ &= \mathbb{I} + ((\mu - \mathbb{I})w)\ (z) + w(z) + \mathcal{C}_-((\mu - \mathbb{I})w)\ (z) + \mathcal{C}_-(w)\ (z) \\ &= m_-(z) + m_-(z)w(z) = m_-(z)v(z) \end{aligned}$$

almost everywhere on $\Sigma\backslash\{0\}$ and the same argument as above gives condition (iv) of the Riemann–Hilbert problem. Finally, uniqueness of solutions can be inferred by a Liouville type procedure (see the proof of Theorem 4.6). □

4.2. The rescaled problem and its approximation

In the following section, the Riemann–Hilbert problem (4.1) will be rescaled in order to obtain a representation that intuitively can be approximated by a t-independent problem. It is then shown, that the jump matrices of the rescaled and the t-independent problem are indeed close to each other and that in general an estimate for a jump matrix leads to an estimate for the solution of a Riemann–Hilbert problem.

We start by introducing rescaled jump matrices via $\hat{v}_j(z) := D(t)^{-1} v_j(zt^{-1/2}) D(t)$, where

$$D(t) = \begin{pmatrix} e^{-it\Theta_0/2} t^{-i\nu/2} & 0 \\ 0 & e^{it\Theta_0/2} t^{i\nu/2} \end{pmatrix}, \qquad t > 0.$$

Explicit formulas for these matrices are given by

$$\hat{v}_1(z) = \begin{pmatrix} 1 & -R_1(zt^{-1/2}) z^{2i\nu} e^{-t(\Theta(zt^{-1/2})-\Theta(0))} \\ 0 & 1 \end{pmatrix},$$
$$\hat{v}_2(z) = \begin{pmatrix} 1 & 0 \\ R_2(zt^{-1/2}) z^{-2i\nu} e^{t(\Theta(zt^{-1/2})-\Theta(0))} & 1 \end{pmatrix},$$
$$\hat{v}_3(z) = \begin{pmatrix} 1 & -R_3(zt^{-1/2}) z^{2i\nu} e^{-t(\Theta(zt^{-1/2})-\Theta(0))} \\ 0 & 1 \end{pmatrix},$$
$$\hat{v}_4(z) = \begin{pmatrix} 1 & 0 \\ R_2(zt^{-1/2}) z^{-2i\nu} e^{t(\Theta(zt^{-1/2})-\Theta(0))} & 1 \end{pmatrix}.$$

A straightforward computation yields that the corresponding rescaled Riemann–Hilbert problem

$$\begin{aligned} &\hat{m}_+(z) = \hat{m}_-(z)\hat{v}_j(z), && z \in \Sigma_j, \quad j = 1,2,3,4, \\ &\hat{m}(z) \to \mathbb{I}, && z \to \infty, \end{aligned} \tag{4.17}$$

is equivalent to the original problem (4.1) in the sense, that whenever m is a solution to (4.1),

$$\hat{m}(z) := D(t)^{-1} m(zt^{-1/2}) D(t)$$

is a solution to (4.17) and whenever $\hat{m}$ solves (4.17),

$$m(z) := D(t)\hat{m}(zt^{1/2})D(t)^{-1}$$

solves (4.1). Remembering the conditions of Theorem 4.1, they already hint that the rescaled problem (4.17) can be replaced by the t-independent problem

$$\begin{aligned} &\hat{m}^c_+(z) = \hat{m}^c_-(z)\hat{v}^c_j(z), && z \in \Sigma_j, \quad j = 1,2,3,4, \\ &\hat{m}^c(z) \to \mathbb{I}, && z \to \infty, \end{aligned} \tag{4.18}$$

corresponding to the jump matrices

$$\hat{v}^c_1(z) = \begin{pmatrix} 1 & -\overline{r_\lambda} z^{2i\nu} e^{-iz^2/2} \\ 0 & 1 \end{pmatrix}, \qquad \hat{v}^c_2(z) = \begin{pmatrix} 1 & 0 \\ r_\lambda z^{-2i\nu} e^{iz^2/2} & 1 \end{pmatrix},$$
$$\hat{v}^c_3(z) = \begin{pmatrix} 1 & -\frac{\overline{r_\lambda}}{1-|r_\lambda|^2} z^{2i\nu} e^{-iz^2/2} \\ 0 & 1 \end{pmatrix}, \quad \hat{v}^c_4(z) = \begin{pmatrix} 1 & 0 \\ \frac{r_\lambda}{1-|r_\lambda|^2} z^{-2i\nu} e^{iz^2/2} & 1 \end{pmatrix}. \tag{4.19}$$

The next lemma assures that a certain kind of approximation indeed is given and even improves for large t.

Lemma 4.4. *The matrices $\hat{v}$ and $\hat{v}^c$ are close in the sense that*

$$\hat{v}(z) = \hat{v}^c(z) + O(\,(L_\lambda + L'_\lambda + |r_\lambda|)\, t^{-\alpha/2} e^{-|z|^2/8}), \qquad z \in \Sigma,$$
$$\hat{v}\,'(z) = \hat{v}^c\,'(z) + O(\,(L_\lambda + L'_\lambda + |r_\lambda|)\, t^{-\alpha/2}(1 + |\log(|z|)|) e^{-|z|^2/8}), \qquad z \in \Sigma,$$

for every $0 < \alpha < 1$, *where the estimates are uniform with respect to* $z \in \Sigma$, $t \geq 1$ *and* $\lambda \in I$.

Proof. To shorten the notation, the index λ appearing in the constants will be suppressed throughout this proof. We only give details for $z \in \Sigma_1$, the other cases being similar. The only nonzero matrix entry in $\hat{v}_j(z) - \hat{v}_j^c(z)$ is the one in the first row and second column given by

$$W = \begin{cases} -R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Theta(zt^{-1/2}) - \Theta(0))} + \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2}, & |z| \leq \rho t^{1/2}, \\ \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} & |z| > \rho t^{1/2}. \end{cases}$$

A straightforward estimate for $|z| \leq \rho t^{1/2}$ shows

$$\begin{aligned} |W| &= \mathrm{e}^{\nu\pi/2} |R_1(zt^{-1/2}) \mathrm{e}^{-t\hat{\Theta}(zt^{-1/2})} - \overline{r}| \mathrm{e}^{-|z|^2/2} \\ &\leq \mathrm{e}^{\nu\pi/2} |R_1(zt^{-1/2}) - \overline{r}| \mathrm{e}^{\mathrm{Re}(-t\hat{\Theta}(zt^{-1/2})) - |z|^2/2} + \mathrm{e}^{\nu\pi/2} |r| |\mathrm{e}^{-t\hat{\Theta}(zt^{-1/2})} - 1| \mathrm{e}^{-|z|^2/2} \\ &\leq \mathrm{e}^{\nu\pi/2} |R_1(zt^{-1/2}) - \overline{r}| \mathrm{e}^{-|z|^2/4} + \mathrm{e}^{\nu\pi/2} |r| t |\hat{\Theta}(zt^{-1/2})| \mathrm{e}^{-|z|^2/4}, \end{aligned}$$

where $\hat{\Theta}(z) = \Theta(z) - \Theta(0) - \frac{\mathrm{i}}{2} z^2$. Here we have used $\frac{\mathrm{i}}{2} z^2 = \frac{1}{2}|z|^2$ for $z \in \Sigma_1$ and $\mathrm{Re}(-t\hat{\Theta}(zt^{-1/2})) \leq |z|^2/4$ by (4.4). Integrating (4.3) gives

$$|R_j(s) - R_j(0)| \leq (L + L')|s| + L'|s| \left|\log(|s|)\right|, \qquad s \in \Sigma_j.$$

Using this and (4.5), we obtain

(4.20)
$$|W| \leq \mathrm{e}^{\nu\pi/2} t^{-\alpha/2} (L + L' + |r| C) \left((1 + K_\alpha)|z| + |z| \left|\log(|z|)\right| + |z|^3 \right) \mathrm{e}^{-|z|^2/4},$$

with $K_\alpha := \sup_{1 \leq s < \infty} s^{\alpha - 1} \log(s)$. For $|z| > \rho t^{1/2}$ we have

$$|W| = |r| \mathrm{e}^{\nu\pi/2} \mathrm{e}^{-|z|^2/2} \leq \mathrm{e}^{\nu\pi/2} |r| \mathrm{e}^{-\rho^2 t/4} \mathrm{e}^{-|z|^2/4},$$

which finishes the proof of the first statement.
Next we turn to estimating the derivatives. Since all other coefficients of the matrix $\hat{v}_j\,(z) - \hat{v}_j^c\,'(z)$ are vanishing, we only have to concern ourselves with W'. First, let $|z| \leq \rho t^{1/2}$. Then

$$\begin{aligned} W'(z) = \quad & -t^{-1/2} R_1'(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Theta(zt^{-1/2}) - \Theta(0))} \\ & + 2\mathrm{i}\nu z^{-1} \left(\overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} - R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Theta(zt^{-1/2}) - \Theta(0))} \right) \\ & + R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Theta(zt^{-1/2}) - \Theta(0))} t^{1/2} \Theta'(zt^{-1/2}) - \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} \mathrm{i}z. \end{aligned}$$

Now, combining assumption (4.3) and previous steps, we can estimate the first term by

$$\begin{aligned} & \left| t^{-1/2} R_1'(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Theta(zt^{-1/2}) - \Theta(0))} \right| \\ & \leq \ \mathrm{e}^{\nu\pi/2} t^{-1/2} (L + L' |\log(|z| t^{-1/2})|) \, \mathrm{e}^{\mathrm{Re}(-t\hat{\Theta}(zt^{-1/2}))} \mathrm{e}^{-|z|^2/2} \\ & \leq \ \mathrm{e}^{\nu\pi/2} t^{-1/2} (L + L' \log(t^{1/2}) + L' \left|\log(|z|)\right|) \, \mathrm{e}^{-|z|^2/4} \\ & \leq \mathrm{e}^{\nu\pi/2} t^{-\alpha/2} \left(L + L' K_\alpha + L' \left|\log(|z|)\right| \right) \mathrm{e}^{-|z|^2/4}. \end{aligned}$$

Concerning the second one, (4.20) gives

$$\left|2\mathrm{i}\nu z^{-1}\right| \left| \overline{r} z^{2\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}z^2/2} - R_1(zt^{-1/2}) z^{2\mathrm{i}\nu} \mathrm{e}^{-t(\Theta(zt^{-1/2}) - \Theta(0))} \right| = 2\nu |z|^{-1} |W(z)|$$

$$\leq 2\nu \mathrm{e}^{\nu\pi/2} t^{-\alpha/2}(L + L' + |r|C)\left(1 + K_\alpha + |\log(|z|)| + |z|^2\right)\mathrm{e}^{-|z|^2/4}.$$

Finally, for the remaining term we have

$$\begin{aligned}
&\left|R_1(zt^{-1/2}) z^{2\mathrm{i}\nu}\mathrm{e}^{-t(\Theta(zt^{-1/2})-\Theta(0))} t^{1/2}\Theta'(zt^{-1/2}) - \overline{r} z^{2\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/2}\mathrm{i}z\right| \\
&\quad = |\hat{v}_{12}(z) t^{1/2}\Theta'(zt^{-1/2}) - \hat{v}_{12}^c(z)\mathrm{i}z| \\
&\quad \leq |\hat{v}_{12}(z) t^{1/2}\Theta'(zt^{-1/2}) - \hat{v}_{12}(z)\mathrm{i}z| + |\hat{v}_{12}(z)\mathrm{i}z - \hat{v}_{12}^c(z)\mathrm{i}z|.
\end{aligned}$$

Assumption (4.6) together with what we have already proven now implies

$$\begin{aligned}
&|\hat{v}_{12}(z) t^{1/2}\Theta'(zt^{-1/2}) - \hat{v}_{12}(z)\mathrm{i}z| \ \leq\ C' t^{-1/2}|z|^2|\hat{v}_{12}(z)| \\
&\quad = C'\mathrm{e}^{\nu\pi/2} t^{-1/2}|z|^2 |R_1(zt^{-1/2})| \mathrm{e}^{\mathrm{Re}(-t\hat{\Theta}(zt^{-1/2}))}\mathrm{e}^{-|z|^2/2} \\
&\quad \leq C'\mathrm{e}^{\nu\pi/2} t^{-1/2}|z|^2 \left(|R_1(zt^{-1/2}) - \overline{r}| + |r|\right)\mathrm{e}^{-|z|^2/4} \\
&\quad \leq C'\mathrm{e}^{\nu\pi/2} t^{-1/2}|z|^2 \left((L+L')t^{-1/2}|z| + L' t^{-1/2}|z|\left|\log(t^{-1/2}|z|)\right| + |r|\right)\mathrm{e}^{-|z|^2/4} \\
&\quad \leq C'\mathrm{e}^{\nu\pi/2} t^{-1/2}|z|^2 \left((L+L')|z| + L' K_0|z| + L'|z|\,|\log(|z|)| + |r|\right)\mathrm{e}^{-|z|^2/4}
\end{aligned}$$

and

$$\begin{aligned}
&|\hat{v}_{12}(z)\mathrm{i}z - \hat{v}_{12}^c(z)\mathrm{i}z| = |z||W(z)| \\
&\leq \mathrm{e}^{\nu\pi/2} t^{-\alpha/2}(L + L' + |r|C)\ \left((1+K_\alpha)|z|^2 + |z|^2\,|\log(|z|)| + |z|^4\right)\ \mathrm{e}^{-|z|^2/4}.
\end{aligned}$$

This means the second statement is proven for $|z| \leq \rho t^{1/2}$. If $|z| \geq \rho t^{1/2}$, one has

$$\begin{aligned}
|W'(z)| \ &=\ |\mathrm{i}z - 2\mathrm{i}\nu z^{-1}||W(z)| \ \leq\ (|z| + 2\nu t^{-1/2}\rho^{-1})|W(z)| \\
&\leq (|z| + 2\nu\rho^{-1})\ \mathrm{e}^{\nu\pi/2}|r|\mathrm{e}^{-\rho^2 t/4}\mathrm{e}^{-|z|^2/4}.
\end{aligned}$$

□

With the help of the following lemma we can transform an estimate for a jump matrix into an estimate for the solution of a RHP.

LEMMA 4.5. *Consider the RHP*

$$\begin{aligned}
m_+(z) &= m_-(z)v(z), & & z \in \Sigma, \\
m(z) &\to \mathbb{I}, & & z \to \infty, \quad z \notin \Sigma.
\end{aligned}$$

Suppose that $\det(v) \equiv 1$, $w = v - \mathbb{I}$ *is continuous on* Σ, *continuously differentiable on* $\Sigma_j\backslash\{0\}$, *satisfies* $w(0) = 0$ *and* $w \in L^\infty(\Sigma) \cap L^2(\Sigma\backslash\{0\})$, $w' \in L^2(\Sigma\backslash\{0\})$. *If* $c, c' > 0$ *denote constants as in Lemma 4.3 and* $c\max\{\|w\|_\infty, \|w'\|_{L^2(\Sigma\backslash\{0\})}\} < 1$, *the Riemann–Hilbert problem has a unique solution* m *and* $\mu = m_-$ *satisfies*

$$\|\mu - \mathbb{I}\|_{H^1(\Sigma\backslash\{0\})} \leq \frac{c'\|w\|_{H^1(\Sigma\backslash\{0\})}}{1 - c\max\{\|w\|_\infty, \|w'\|_{L^2(\Sigma\backslash\{0\})}\}}.$$

PROOF. The statement is a direct consequence of Lemma 4.3, if we use the Neumann series representation of $(\mathrm{id} - C_w)$. □

4.3. Asymptotics for the time-independent problem

Next, we find the asymptotic behavior of the solution to the Riemann–Hilbert problem (4.18). The proof for this result is well-known - in this thesis, we follow its presentation in [**14**], but also include an instructive motivation that can be found for example in [**12**]. Together with the above procedure this will in the end enable us to find the asymptotics for the solution of the rescaled problem.

THEOREM 4.6. *Assume* $r \in \mathbb{D}$ *and consider the Riemann–Hilbert problem* (4.18)*, where the jump matrices* $\hat{v}_j^c$*,* $j = 1, ..., 4$*, are defined by* (4.19) *with* r_λ *replaced by* r *and* $\nu = -\frac{1}{2\pi}\log(1-|r|^2)$*. Then this problem has a unique solution* $\hat{m}^c$ *that can be represented as*

$$\hat{m}^c(z) = \mathbb{I} + \frac{1}{z}\hat{M}^c + O\left(\frac{1}{z^2}\right),$$

where

$$\hat{M}^c = \mathrm{i}\begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix}, \quad \beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))} \tag{4.21}$$

and the error estimate holds uniformly for $z \in \mathbb{C}\backslash\Sigma$*. It is uniform with respect to* r *in compact subsets of* $\mathbb{D}$ *as well. Moreover, the solution* $\hat{m}^c$ *is bounded on the whole of* $\mathbb{C}\backslash\Sigma$*. The bound can again be chosen uniformly for* r *in compacts subsets of* $\mathbb{D}$*.*

Since the proof is fairly long, we will first give a brief outline and motivate the ansatz we use to construct the solution. As this is only supposed to provide an overall idea of the actual proof, no detailed calculations are performed here.

Assume $\hat{m}^c(z)$ is the solution of the Riemann–Hilbert problem (4.18). In order to remove the jump on Σ, we denote the region enclosed by $\mathbb{R}$ and Σ_j as Ω_j (cf. Figure 4.4) and set

$$\psi(z) = \hat{m}^c(z)\begin{cases} T_0(z)T_j, & z \in \Omega_j,\ j = 1, \dots, 4, \\ T_0(z), & \text{else}, \end{cases} \tag{4.22}$$

where

$$T_0(z) = \begin{pmatrix} z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4} & 0 \\ 0 & z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4} \end{pmatrix},$$

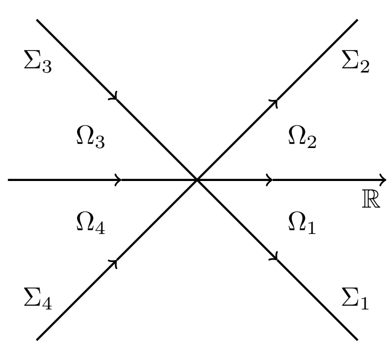


FIGURE 4.4. The regions of $\mathbb{C}\backslash\Sigma$.

and

$$T_1 = \begin{pmatrix} 1 & \overline{r} \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & -\frac{\overline{r}}{1-|r|^2} \\ 0 & 1 \end{pmatrix}, \quad T_4 = \begin{pmatrix} 1 & 0 \\ -\frac{r}{1-|r|^2} & 1 \end{pmatrix}.$$

Then ψ turns out to be homolorphic on $\mathbb{C}\backslash\mathbb{R}$ and satisfy the jump condition

$$\psi_+(z) = \psi_-(z) \begin{pmatrix} 1-|r|^2 & -\overline{r} \\ r & 1 \end{pmatrix}, \qquad z \in \mathbb{R}. \tag{4.23}$$

Up to this point, our considerations have been rigorous. But as our next aim is only to motivate the later use of (4.24), we will now switch to proceeding in an informal manner and also use steps that are not fully justified. Taking into account that the jump matrix for ψ is constant, $\frac{d}{dz}\psi$ satisfies the jump condition (4.23), too. Assuming we already know that $\hat{m}^c$ satisfies the claimed asymptotics, we have for $\frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}$ and $|z|$ sufficiently large

$$\begin{aligned} &\left(\frac{d}{dz}\psi(z) + \frac{\mathrm{i}z}{2}\sigma_3\psi(z)\right)\psi^{-1}(z) = \\ &= \left(\mathrm{i}\left(\frac{\nu}{z} - \frac{z}{2}\right)\hat{m}^c(z)\sigma_3 + \frac{d}{dz}\hat{m}^c(z) + \frac{\mathrm{i}z}{2}\sigma_3\hat{m}^c(z)\right)\hat{m}^c(z)^{-1} \\ &= B + O\left(\frac{1}{z}\right), \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \beta \\ \overline{\beta} & 0 \end{pmatrix}. \end{aligned}$$

The left hand side of the equation is entire and using Liouville's theorem, we obtain the differential equation

$$\frac{d}{dz}\psi(z) + \frac{\mathrm{i}z}{2}\sigma_3\psi(z) = B\psi(z). \tag{4.24}$$

In the proof of the theorem, the above procedure will somehow be reversed. Using parabolic cylinder functions, we can explicitly construct a special function ψ which satisfies (4.24). From this we prove that ψ satisfies the jump condition in (4.23). Next, it is shown that by inverting the above transformation we obtain a function $\hat{m}^c$ on $\mathbb{C}\backslash\Sigma$ that is a solution to the jump condition of (4.18). Then we show that this newly constructed $\hat{m}^c$ is bounded and satisfies the asymptotics described in the theorem. This is possible due to the fact that the asymptotics for the parabolic cylinder functions are known. In particular we thus have that $\hat{m}^c$ is a solution to the full Riemann–Hilbert problem (4.18), which means we are done.

Proof. The determinant of the jump matrix $\hat{v}^c$ is equal to 1, so uniqueness of solutions to the (4.18) can be obtained applying the usual Liouville argument. To be precise, suppose m is a solution to (4.18). Then $\det(m)$ is holomorphic on $\mathbb{C}\backslash\Sigma$ and continuously extendable to the whole of $\mathbb{C}\backslash\{0\}$, since on $\Sigma\backslash\{0\}$ we have

$$(\det(m))_+ = \det(m_+) = \det(m_-\hat{v}^c) = \det(m_-) = (\det(m))_-.$$

Now Morera's theorem yields that $\det(m)$ is holomorphic on $\mathbb{C}\backslash\{0\}$. Because $\det(m)$ is bounded near the origin, it can be extended to an entire function by Riemann's theorem. But $\lim_{z\to\infty}\det(m(z)) = 1$, hence Liouville's theorem gives $\det(m) \equiv 1$ on $\mathbb{C}$. Now assume $n(z)$ is another solution to (4.18). By the above argument, n is invertible on $\mathbb{C}\backslash\Sigma$, so we can consider $j(z) = m(z)n^{-1}(z)$ on $\mathbb{C}\backslash\Sigma$. Then

$$j_+ = m_+(n_+)^{-1} = m_-\hat{v}^c(n_-\hat{v}^c)^{-1} = m_-n_- = j_-,$$

so the above procedure leads to $j \equiv \mathbb{I}$.

For $a \in \mathbb{C}$, let D_a be the entire parabolic cylinder function from §16.5 in [**23**] or chapter 12 in [**19**]. Now for $z \in \mathbb{C}\backslash\mathbb{R}$ set

$$\psi(z) = \begin{pmatrix} \psi_{11}(z) & \frac{1}{\beta}\left(\frac{d}{dz} - \frac{iz}{2}\right)\psi_{22}(z) \\ \frac{1}{\beta}\left(\frac{d}{dz} + \frac{iz}{2}\right)\psi_{11}(z) & \psi_{22}(z) \end{pmatrix},$$

where

$$\psi_{11}(z) = \begin{cases} \mathrm{e}^{-3\pi\nu/4} D_{\mathrm{i}\nu}\left(\mathrm{e}^{-3\mathrm{i}\pi/4} z\right), & \operatorname{Im} z > 0, \\ \mathrm{e}^{\pi\nu/4} D_{\mathrm{i}\nu}\left(\mathrm{e}^{\mathrm{i}\pi/4} z\right), & \operatorname{Im} z < 0, \end{cases}$$

$$\psi_{22}(z) = \begin{cases} \mathrm{e}^{\pi\nu/4} D_{-\mathrm{i}\nu}\left(\mathrm{e}^{-\mathrm{i}\pi/4} z\right), & \operatorname{Im} z > 0, \\ \mathrm{e}^{-3\pi\nu/4} D_{-\mathrm{i}\nu}\left(\mathrm{e}^{3\mathrm{i}\pi/4} z\right), & \operatorname{Im} z < 0. \end{cases}$$

By the recursion formula (see [**19**])

$$\frac{d}{dz} D_a(z) = a D_{a-1}(z) - \frac{z}{2} D_a(z), \tag{4.25}$$

the off-diagonal entries can also be written as

$$\psi_{12}(z) = \begin{cases} \beta \mathrm{e}^{\pi\nu/4} \mathrm{e}^{-3\mathrm{i}\pi/4} D_{-\mathrm{i}\nu-1}(\mathrm{e}^{-\mathrm{i}\pi/4} z), & \operatorname{Im} z > 0, \\ \beta \mathrm{e}^{-3\pi\nu/4} \mathrm{e}^{\mathrm{i}\pi/4} D_{-\mathrm{i}\nu-1}(\mathrm{e}^{3\mathrm{i}\pi/4} z), & \operatorname{Im} z < 0, \end{cases}$$

$$\psi_{21}(z) = \begin{cases} \overline{\beta} \mathrm{e}^{-3\pi\nu/4} \mathrm{e}^{-\mathrm{i}\pi/4} D_{\mathrm{i}\nu-1}(\mathrm{e}^{-3\mathrm{i}\pi/4} z), & \operatorname{Im} z > 0, \\ \overline{\beta} \mathrm{e}^{\nu\pi/4} \mathrm{e}^{3\mathrm{i}\pi/4} D_{\mathrm{i}\nu-1}(\mathrm{e}^{\mathrm{i}\pi/4} z), & \operatorname{Im} z < 0. \end{cases}$$

In the following we will show that $\hat{m}^c$, defined on $z \in \mathbb{C}\backslash(\Sigma \cup \mathbb{R})$ by

$$\hat{m}^c(z) = \psi(z) \begin{cases} T_j^{-1} T_0^{-1}(z), & z \in \Omega_j,\ j = 1, \ldots, 4, \\ T_0^{-1}(z), & \text{else}, \end{cases}$$

is the solution to (4.18).

We start by proving that ψ satisfies the jump condition (4.23). The parabolic cylinder function D_a satisfies $D_a''(\zeta) + (a + \frac{1}{2} - \frac{1}{4}\zeta^2) D_a(\zeta) = 0$, which yields (4.24). Moreover, for the limits ψ_+ and ψ_- the corresponding ordinary differential equation $m'(x) + \frac{\mathrm{i}x}{2}\sigma_3 m(x) = B m(x)$ holds. Writing everything component by component gives that

$$\begin{pmatrix} \psi_{11,+} \\ \psi_{21,+} \end{pmatrix}, \qquad \begin{pmatrix} \psi_{12,+} \\ \psi_{22,+} \end{pmatrix}, \qquad \begin{pmatrix} \psi_{11,-} \\ \psi_{21,-} \end{pmatrix}, \qquad \begin{pmatrix} \psi_{12,-} \\ \psi_{22,-} \end{pmatrix}$$

are solutions to the following linear system of ordinary differential equations:

$$y_1'(x) + \frac{\mathrm{i}x}{2} y_1(x) - \beta y_2(x) = 0,$$
$$y_2'(x) - \frac{\mathrm{i}x}{2} y_2(x) - \overline{\beta} y_1(x) = 0.$$

Since there are exactly two linearly independent solutions, there must be a constant matrix v such that $\psi_+ = \psi_- v$. A straightforward computation gives

$$v = \psi_-(0)^{-1} \psi_+(0) = \begin{pmatrix} \mathrm{e}^{-3\pi\nu/4} D_{-\mathrm{i}\nu}(0) & -\frac{\mathrm{e}^{-3\pi\nu/4}\mathrm{e}^{3\mathrm{i}\pi/4}}{\overline{\beta}} D'_{-\mathrm{i}\nu}(0) \\ -\frac{\mathrm{e}^{\pi\nu/4}\mathrm{e}^{\mathrm{i}\pi/4}}{\beta} D'_{\mathrm{i}\nu}(0) & \mathrm{e}^{\pi\nu/4} D_{\mathrm{i}\nu}(0) \end{pmatrix}$$

$$\cdot \begin{pmatrix} \mathrm{e}^{-3\pi\nu/4} D_{\mathrm{i}\nu}(0) & \frac{\mathrm{e}^{\pi\nu/4}\mathrm{e}^{-\mathrm{i}\pi/4}}{\overline{\beta}} D'_{-\mathrm{i}\nu}(0) \\ \frac{\mathrm{e}^{-3\pi\nu/4}\mathrm{e}^{-3\mathrm{i}\pi/4}}{\beta} D'_{\mathrm{i}\nu}(0) & \mathrm{e}^{\pi\nu/4} D_{-\mathrm{i}\nu}(0) \end{pmatrix} = \begin{pmatrix} 1-|r|^2 & -\overline{r} \\ r & 1 \end{pmatrix},$$

where the following formulas (see [**19**]) have been used:

$$D_a(0) = \frac{2^{\frac{a}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-a}{2}\right)}, \qquad D'_a(0) = -\frac{2^{\frac{1+a}{2}}\sqrt{\pi}}{\Gamma\left(-\frac{a}{2}\right)},$$

$$\Gamma(z)z = \Gamma(1+z), \qquad |\Gamma(-\mathrm{i}x)|^2 = \frac{\Gamma(1-\mathrm{i}x)\Gamma(\mathrm{i}x)}{-\mathrm{i}x} = \frac{\pi}{\nu\sinh(\pi x)},$$

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \qquad \Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

Next, we will prove that $\hat{m}^c$ can be extended to a holomorphic function on $\mathbb{C}\backslash\Sigma$ and satisfies the jump condition of (4.18) on Σ. Indeed, the jump along $\mathbb{R}$ is vanishing, as due to our previous steps we have

$$\begin{aligned} \hat{m}^c_+(z) = \psi_+(z)T_2^{-1}T_0^{-1}(z) &= \psi_-(z)\begin{pmatrix} 1-|r|^2 & -\overline{r} \\ r & 1 \end{pmatrix} T_2^{-1}T_0^{-1}(z) \\ &= \psi_-(z)T_1^{-1}T_0^{-1}(z) = \hat{m}^c_-(z), \qquad z>0, \end{aligned}$$

and

$$\begin{aligned} \hat{m}^c_+(z) = \psi_+(z)T_3^{-1}T_{0,+}^{-1}(z) &= \psi_-(z)\begin{pmatrix} 1-|r|^2 & -\overline{r} \\ r & 1 \end{pmatrix} T_3^{-1}T_{0,+}^{-1}(z) \\ &= \psi_-(z)T_4^{-1}T_{0,-}^{-1}(z) = \hat{m}^c_-(z), \qquad z<0. \end{aligned}$$

Here, $T_{0,\pm}$ denotes the limit of $T_0(z)$ taken from the left resp. right side onto the negative real axis, which due to our choice of the branch of the logarithm is given by

$$T_{0,\pm}(z) = \begin{pmatrix} \mathrm{e}^{\mp\pi\nu}|z|^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4} & 0 \\ 0 & \mathrm{e}^{\pm\pi\nu}|z|^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4} \end{pmatrix}, \qquad z<0.$$

Applying Morera's theorem gives holomorphicity. To verify the jump condition on, say Σ_1, we see that

$$\hat{m}^c_+(z) = \psi(z)T_1^{-1}(z)T_0^{-1}(z) = \psi(z)T_0^{-1}(z)\,T_0(z)T_1^{-1}(z)T_0^{-1}(z) = \hat{m}^c_-(z)\hat{v}^c_1(z).$$

Repeating this calculation for the other segments of Σ proves the claim.

All that is left now is to verify that $\hat{m}^c$ behaves asymptotically as claimed in the theorem and is bounded uniformly for r in a compact subset of $\mathbb{D}$. We know (see §16.5 and §16.52 in [**23**]) that for every fixed $\delta>0$, we have for $z\to\infty$ the asymptotic formula

$$D_a(z) = z^a\mathrm{e}^{-z^2/4}\left(1-\frac{a(a-1)}{2z^2}+O\left(z^{-4}\right)\right)+$$

$$\begin{cases} 0, & \arg(z)\in[-\frac{3\pi}{4}+\delta,\frac{3\pi}{4}-\delta], \\ -\mathrm{e}^{\mathrm{i}\pi a}\frac{\sqrt{2\pi}\mathrm{e}^{z^2/4}z^{-a-1}}{\Gamma(-a)}\left(1+\frac{(a+1)(a+2)}{2z^2}+O\left(z^{-4}\right)\right), & \arg(z)\in[\frac{\pi}{4}+\delta,\frac{5\pi}{4}-\delta], \\ -\mathrm{e}^{-\mathrm{i}\pi a}\frac{\sqrt{2\pi}\mathrm{e}^{z^2/4}z^{-a-1}}{\Gamma(-a)}\left(1+\frac{(a+1)(a+2)}{2z^2}+O\left(z^{-4}\right)\right), & \arg(z)\in[-\frac{5\pi}{4}+\delta,-\frac{\pi}{4}-\delta]. \end{cases}$$

Here, all error terms are uniform with respect to a in compact subsets of $\mathbb{C}$. This means, we can obtain asymptotic expansions for ψ and $\hat{m}^c$ in the different regions of $\mathbb{C}\backslash\mathbb{R}$ (see figure Figure 4.1). Since the computation is long, but straightforward, we will only deal with the region Ω_2 here. In the following, all asymptotics will hold for $z\to\infty, z\in\Omega_2$ and all appearing error terms will be uniform with respect to $\arg(z)$ and r in compact subsets of $\mathbb{D}$. By the asymptotics for $D_a(z)$,

$$\psi_{11}(z) = z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}(1+O(z^{-2})) - \frac{\sqrt{2\pi}}{\Gamma(-\mathrm{i}\nu)}\mathrm{e}^{-\pi\nu/2}\mathrm{e}^{3\mathrm{i}\pi/4}z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}\left(\frac{1}{z}+O(z^{-3})\right),$$
$$\psi_{12}(z) = -\mathrm{i}\beta z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}\left(\frac{1}{z}+O(z^{-3})\right),$$
$$\psi_{21}(z) = \mathrm{i}\overline{\beta}z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}\left(\frac{1}{z}+O(z^{-3})\right) - \frac{\sqrt{2\pi}\mathrm{e}^{-\pi\nu/2}\mathrm{e}^{-3\mathrm{i}\pi/4}}{\beta\Gamma(-\mathrm{i}\nu)}z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}\left(1+O(z^{-2})\right),$$
$$\psi_{22}(z) = z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(1+O(z^{-2})).$$

Finally, we can use

$$\beta = \frac{\sqrt{2\pi}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}}{\Gamma(-\mathrm{i}\nu)r} \quad \text{and} \quad \left|\mathrm{e}^{\mathrm{i}z^2/4}\right| \le 1 \text{ for } z\in\Omega_2$$

to get

$$\begin{aligned}
\hat{m}^c(z) &= \psi(z)T_1^{-1}T_0^{-1}(z)\\
&= \begin{pmatrix} z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}(1+O(z^{-2})) & -\mathrm{i}\beta z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}\left(\frac{1}{z}+O(z^{-2})\right)\\ \mathrm{i}\overline{\beta}z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}\left(\frac{1}{z}+O(z^{-2})\right) & z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(1+O(z^{-2}))\end{pmatrix}T_0^{-1}(z)\\
&\quad+\begin{pmatrix} \frac{\sqrt{2\pi}}{\Gamma(-\mathrm{i}\nu)}\mathrm{e}^{3\mathrm{i}\pi/4}\mathrm{e}^{-\pi\nu/2}z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}\left(\frac{1}{z}-\frac{1}{z}+O(z^{-2})\right) & 0\\ rz^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(1-1+O(z^{-2})) & 0\end{pmatrix}T_0^{-1}(z)\\
&= \begin{pmatrix} z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}(1+O(z^{-2})) & -\mathrm{i}\beta z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}\left(\frac{1}{z}+O(z^{-2})\right)\\ \mathrm{i}\overline{\beta}z^{\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}z^2/4}\left(\frac{1}{z}+O(z^{-2})\right) & z^{-\mathrm{i}\nu}\mathrm{e}^{\mathrm{i}z^2/4}(1+O(z^{-2}))\end{pmatrix}T_0^{-1}(z)\\
&= \begin{pmatrix} 1+O(z^{-2}) & -\mathrm{i}\beta\frac{1}{z}+O(z^{-2})\\ \mathrm{i}\overline{\beta}\frac{1}{z}+O(z^{-2}) & 1+O(z^{-2})\end{pmatrix}.
\end{aligned}$$

Repeating this step for the other regions of $\mathbb{C}\backslash\Sigma$ we obtain the claimed asymptotic behavior. Taking into account that D_a is entire also as a function of a (see [**19**]) and that $T_0(z)$, restricted to a compact set, is bounded uniformly with respect to ν in a compact set, the theorem is proven. □

4.4. Transferring the asymptotics back to the original problem

Upon comparing the rescaled problem to the t-independent problem, we can find the asypmtotics for the rescaled problem. The asymptotics for the original problem are then obtained by simply reversing the scaling process.

Lemma 4.7. *There is some $T>0$, such that for every $t\ge T$ and $\lambda\in I$, the Riemann–Hilbert problem* (4.17) *is uniquely solvable and the solution $\hat{m}(z)$ has an asymptotic expansion*

$$\hat{m}(z) = \mathbb{I} + \frac{1}{z}\hat{M}^c + \frac{1}{z}\hat{e}(z) + \hat{h}(z), \tag{4.26}$$

where $\hat{M}^c$ is defined by (4.21) *and the error terms $\hat{e}(z)$ and $\hat{h}(z)$ are of order*

$$\hat{e}(z) = O\Big(\,(L_\lambda + L'_\lambda + |r_\lambda|)\;t^{-\alpha/2}\Big) \text{ for every } 0<\alpha<1 \quad\text{and}\quad \hat{h}(z) = O\left(\frac{1}{z^2}\right)$$

uniformly for $z \in \mathbb{C}\backslash\Sigma$, $t \geq T$ *and* $\lambda \in I$.

Proof. Consider the auxiliary RHP corresponding to the jump matrix

$$\hat{v}^d(z) = \hat{m}^c_-(z)\hat{v}(z)\hat{v}^c(z)^{-1}\hat{m}^c_-(z)^{-1} = \mathbb{I} + \hat{m}^c_-(z)\big(\hat{v}(z) - \hat{v}^c(z)\big)\hat{m}^c_-(z)^{-1}.$$

Then the problem with jump matrix $\hat{v}$ is solvable if and only if the problem with jump matrix $\hat{v}^d$ is solvable. In this case, the unique solutions are connected by the formula $\hat{m}^d(z) = \hat{m}(z)\hat{m}^c(z)^{-1}$. Establishing estimates for $\hat{v}^d$, the boundedness of $\hat{m}^c$ implies

$$\hat{w}^d(s) = O(t^{-\alpha/2}\ (L_\lambda + L'_\lambda + |r_\lambda|)\mathrm{e}^{-|s|^2/8})$$

for every $0 < \alpha < 1$. If we write $\hat{m}^c$ in terms of the parabolic cylinder functions D_a (see the proof of Theorem 4.6), combine estimate (4.20), the analycity of $D_a(s)$ in both a and s (see [**19**]), and that (see §16.5 and §16.52 in [**23**]) for $x \in \mathbb{R}, x \to \infty$ we have the asymptotic formula

$$D_a(x) = x^a \mathrm{e}^{-x^2/4} O(1)$$

with the error uniform for a in compact subsets of $\mathbb{C}$, we obtain

$$\hat{w}^{d\,\prime}(s) = O(t^{-\alpha/2}\ (L_\lambda + L'_\lambda + |r_\lambda|)\,(1 + |\log(|s|)|)\,\mathrm{e}^{-|s|^2/16})$$

for every $0 < \alpha < 1$. Hence, for t is sufficiently large, Lemmas 4.3 and 4.5 yield the existence of a unique solution $\hat{m}^d$ with

$$\hat{m}^d(z) = \ \mathbb{I} - \frac{1}{z}\frac{1}{2\pi\mathrm{i}}\int_\Sigma \hat{\mu}^d(s)\hat{w}^d(s)ds + \frac{1}{z}\frac{1}{2\pi\mathrm{i}}\int_\Sigma s\hat{\mu}^d(s)\hat{w}^d(s)\frac{ds}{s-z}, \quad z \in \mathbb{C}\backslash\Sigma,$$

where $\mu = \hat{m}^d_-$ and

$$\|\hat{\mu}^d - \mathbb{I}\|_{H^1(\Sigma\backslash\{0\})} = O(t^{-\alpha/2}\ (L_\lambda + L'_\lambda + |r_\lambda|))$$

for every $0 < \alpha < 1$. The first integral is clearly of order $t^{-\alpha/2}(L_\lambda + L'_\lambda + |r_\lambda|)$. Now, let $\chi \in \mathcal{C}^\infty_0(\mathbb{R}^2), \chi \equiv 1$ for $|s| \leq 1$, $\chi \equiv 0$ for $|s| \geq 2$ and set $\chi_z(s) = \chi(4s/|z|)$. Splitting the second integral in three terms, we see

$$\frac{1}{2\pi\mathrm{i}}\int_\Sigma s\chi_z(s)(\hat{\mu}^d(s) - \mathbb{I})\hat{w}^d(s)\frac{ds}{s-z} = O\left(\frac{1}{z}\right),$$

by using that $|z| \leq 2|s-z|$ within the support of the integrand, and

$$\frac{1}{2\pi\mathrm{i}}\int_\Sigma s(1-\chi_z(s))(\hat{\mu}^d(s) - \mathbb{I})\hat{w}^d(s)\frac{ds}{s-z} = O(t^{-\alpha/2}\ (L_\lambda + L'_\lambda + |r_\lambda|)),$$

$$\frac{1}{2\pi\mathrm{i}}\int_\Sigma s\hat{w}^d(s)\frac{ds}{s-z} = O(t^{-\alpha/2}\ (L_\lambda + L'_\lambda + |r_\lambda|)),$$

by applying (4.11), the Sobolev inequality (4.8) and the inequality $2|s| \leq |z|$ for s in the support of χ_z. The conclusion now follows from Theorem 4.6. □

Lemma 4.7 finally implies Theorem 4.1, as

$$m(z) = D(t)\hat{m}(zt^{1/2})D(t)^{-1}$$

$$= \mathbb{I} + \frac{1}{t^{1/2}z}D(t)\hat{M}^c D(t)^{-1} + \frac{1}{t^{1/2}z}D(t)\hat{e}(zt^{1/2})D(t)^{-1} + D(t)\hat{h}(zt^{1/2})D(t)^{-1}$$

and $D(t)$ stays bounded.

CHAPTER 5

The asymptotics in the similarity region

After these preparations, we are ready to prove this thesis' main result. In this chapter, we obtain an asymptotic expansion for the solution of the Korteweg–de Vries equation in the similarity region. This will be done by deriving an expansion for the solution $\hat{m}$ of the transformed Riemann–Hilbert problem and then using the relations from Section 2.1.

5.1. The similarity region and some basic estimates

This section will provide us with the definition of the similarity region and a few important uniform estimates.

First of all, we fix some notation and set $S_C = \{(x,t) \in \mathbb{R} \times [0,\infty);\ x \le -Ct\}$ for $C > 0$. Then S_C will be called the similarity region for C. In this context, we also write $C_0 := \sqrt{C/12}$.
Next, we deal with a minor technical issue occuring in the definition of the jump contour $\hat{\Sigma}$. There is still some freedom in the construction suggested by Figure 3.2. Namely, we did not specify the distance of the real axis to the parts of $\hat{\Sigma}(x,t)$ that are parallel to it. This will now be helped by a precise assumption. Whenever a similarity region S_C is considered, the jump contour $\hat{\Sigma}(x,t)$ for a pair $(x,t) \in S_C$ will be the one defined by Figure 3.2 with the distance of the real axis to the parallel parts equal to $\delta_C := \min\{\delta_R/2,\ C_0/2,\ (\kappa_1 - \check{\epsilon})/2\}$. With this convention, we can state and prove the next lemma. It will finally clarify the way, in which $\hat{v}(x,t)$ converges to the identity for $t \to \infty$.

LEMMA 5.1. *Assume a similarity region S_C and let $\epsilon > 0$, $1 \le p \le \infty$ be two parameters. For $k \in \mathbb{R}$, denote by $U_\epsilon(k) = \overline{D(k,\epsilon)} \cup \overline{D(-k,\epsilon)}$ the union of the two closed discs with radius ϵ centered at the points $\pm k$. Then there exist positive constants K, a and b such that*

$$\|\hat{v}(x,t) - \mathbb{I}\|_{L^p(\hat{\Sigma}(x,t)\backslash U_\epsilon(k_0))} \ \le\ K\mathrm{e}^{-at-bk_0}$$

for every $(x,t) \in S_C$ with $t \ge 1$.

PROOF. First of all, notice that

$$\left| \int_{-k_0}^{k_0} \log\left(|T(\zeta)|^2\right) \frac{d\zeta}{\zeta - k} \right| \ \le\ \frac{1}{|\operatorname{Im}(k)|} \| \log\left(|T(\cdot)|^2\right) \|_{L^1(\mathbb{R})}, \qquad k \in \mathbb{C}\backslash\mathbb{R}.$$

Therefore, all factors in the nonzero entry of $\hat{v}(x,t) - \mathbb{I}$ except the exponential one are bounded on $\hat{\Sigma}(x,t)\backslash U_\epsilon(k_0)$ uniformly for $(x,t) \in S_C$. Thus, we only need to consider the L^p norm of the remaining factor. The lemma then follows from

$$t\operatorname{Re}(\Phi(k,x,t)) = 8tb^3 - 24ta^2b - 2xb, \qquad k \in \mathbb{C},\ k = a + \mathrm{i}b,$$

and a straightforward estimate. □

Next, we investigate the exponent $\psi(k, k_0)$ defined in Lemma 3.4 more closely and analyze its growth near the critical point k_0. It turns out, that in some way, the behavior is uniform with respect to similarity regions as well.

LEMMA 5.2. *Assume a similarity region S_C. Then there is some constant $K > 0$, such that*

$$|\psi(k,k_0)| \le \frac{K}{k_0} \quad \textit{and} \quad |\psi'(k,k_0)| \le K + \frac{K}{k_0^4}|\log(|k-k_0|)| \tag{5.1}$$

for $(x,t) \in S_C$ and $k \in k_0 + \Sigma\backslash\{0\}$ with $|k - k_0| \le \frac{C_0}{8}$.

PROOF. We start by providing growth rates for the function $\phi(s) := \log(|T(s)|^2)$, $s \in \mathbb{R}\backslash\{0\}$. By the results in Section 2.1, $\phi(s) \in L^1(\mathbb{R})$ and

$$\phi(s) = O\left(\frac{1}{s^8}\right), \quad s \in \mathbb{R},\ |s| \to \infty.$$

In view of our assumptions on the reflection coefficient and Cauchy's inequality, the derivatives $R'(k)$ and $R''(k)$ are bounded for $k \in \mathbb{R}$. This leads to

$$\phi'(s) = 2\frac{\mathrm{Re}(\overline{R(s)}R'(s))}{|R(s)|^2-1} = O\left(\frac{1}{s^4}\right), \quad s \in \mathbb{R},\ |s| \to \infty,$$

and

$$\phi''(s) = 2\frac{\mathrm{Re}(\overline{R'(s)}R'(s) + \overline{R(s)}R''(s))}{|R(s)|^2-1} - \phi'(s)^2 = O(1), \quad s \in \mathbb{R},\ |s| \to \infty.$$

To prove the estimate for $|\psi(k,k_0)|$, we write

$$\psi(k,k_0) = \frac{1}{2\pi\mathrm{i}}\int_{-k_0}^{\frac{k_0}{2}} \frac{\phi(\zeta)-\phi(k_0)}{\zeta-k}\,d\zeta + \frac{1}{2\pi\mathrm{i}}\int_{\frac{k_0}{2}}^{k_0} \frac{\phi(\zeta)-\phi(k_0)}{\zeta-k}\,d\zeta.$$

For $\zeta \in [-k_0, k_0/2]$, we have $|\zeta - k| \ge 3k_0/8$, so

$$\begin{aligned}\left|\frac{1}{2\pi\mathrm{i}}\int_{-k_0}^{\frac{k_0}{2}} \frac{\phi(\zeta)-\phi(k_0)}{\zeta-k}\,d\zeta\right| &\le \frac{4}{3\pi}\frac{1}{k_0}\int_{-k_0}^{\frac{k_0}{2}} |\phi(\zeta)| + |\phi(k_0)|\,d\zeta \\ &\le \frac{4}{3\pi}\frac{1}{k_0}\left(\|\phi\|_{L^1(\mathbb{R})} + \frac{3}{2}k_0|\phi(k_0)|\right).\end{aligned}$$

This means that the desired estimate holds for the first integral. Dealing with the second one, there is a constant $L > 0$, such that $|\phi'(s)| \le Ls^{-4}$ for $s \ge C_0/2$. Hence,

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{\frac{k_0}{2}}^{k_0} \frac{\phi(\zeta)-\phi(k_0)}{\zeta-k}\,d\zeta\right| \le \frac{8L}{\pi k_0^4}\int_{\frac{k_0}{2}}^{k_0} \frac{|\zeta-k_0|}{|\zeta-k|}\,d\zeta \le \frac{8L}{\pi k_0^4}\left(\frac{k_0}{2} + \int_{\frac{k_0}{2}}^{k_0} \frac{|k-k_0|}{|\zeta-k|}\,d\zeta\right)$$

by the mean value theorem. Since $|\mathrm{Im}(k-k_0)| = |\mathrm{Re}(k-k_0)| = |\mathrm{Im}(k)|$, this is again smaller than

$$\frac{8L}{\pi k_0^4}\left(\frac{k_0}{2} + \int_{\frac{k_0}{2}}^{k_0} \frac{\sqrt{2}\,|\mathrm{Im}(k)|}{|\mathrm{Im}(k)|}\,d\zeta\right) = \frac{8L}{\pi k_0^4}\left(\frac{k_0}{2} + \frac{k_0}{\sqrt{2}}\right).$$

The first claim is proven. In order to obtain the logarithmic estimate for the derivative, we use partial integration to see

$$\psi'(k,k_0) = \frac{1}{2\pi \mathrm{i}} \int_{-k_0}^{k_0} \frac{\phi(\zeta)-\phi(k_0)}{(\zeta-k)^2}\, d\zeta$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{-k_0}^{\frac{k_0}{2}} \frac{\phi(\zeta)-\phi(k_0)}{(\zeta-k)^2}\, d\zeta + \frac{1}{2\pi \mathrm{i}} \frac{\phi\left(\frac{k_0}{2}\right)-\phi(k_0)}{\frac{k_0}{2}-k} + \frac{1}{2\pi \mathrm{i}} \int_{k_0-\frac{C_0}{4}}^{k_0} \frac{\phi'(\zeta)-\phi'(k_0)}{\zeta-k}\, d\zeta$$

$$+\frac{1}{2\pi \mathrm{i}} \int_{\frac{k_0}{2}}^{k_0-\frac{C_0}{4}} \frac{\phi'(\zeta)}{\zeta-k}\, d\zeta + \frac{\phi'(k_0)}{2\pi \mathrm{i}} \int_{k_0-\frac{C_0}{4}}^{k_0} \frac{1}{\zeta-k}\, d\zeta$$

$$= T_1 + T_2 + T_3 + T_4 + T_5.$$

Using the same steps as before, we can show

$$|T_1| \;\le\; \frac{32}{9\pi}\frac{1}{k_0^2}\left(\|\phi\|_{L^1(\mathbb{R})} + \frac{3}{2}k_0|\phi(k_0)|\right).$$

But this means that the first term can be estimated in the way claimed in the lemma. The same holds for the second term, since $|k_0/2-k| \ge 3k_0/8$ and therefore

$$|T_2| \;\le\; \frac{4}{3\pi}\frac{1}{k_0}(|\phi(k_0/2)| + |\phi(k_0)|).$$

Next, we deal with the third term. There is a constant $L' > 0$ such that $|\phi''(s)| \le L'$ for all $s \ge C_0$. If we use the mean value theorem and proceed as in the first part of the proof, we get

$$|T_3| \;\le\; \frac{1}{2\pi}L' \int_{k_0-\frac{C_0}{4}}^{k_0} \frac{|\zeta-k_0|}{|\zeta-k|}\, d\zeta$$

$$\le\; \frac{1}{2\pi}L'\left(\frac{C_0}{4} + \int_{k_0-\frac{C_0}{4}}^{k_0} \frac{|k-k_0|}{|\zeta-k|}\, d\zeta\right) \;\le\; \frac{1}{2\pi}L'\left(\frac{C_0}{4} + \frac{\sqrt{2}C_0}{4}\right).$$

By the above, we have $|\phi'(\zeta)| \le L16/k_0^4$ and $|\zeta-k| \ge C_0/8$ for $\zeta \in [k_0/2, k_0-C_0/4]$. This leads to an estimate for the fourth term given by

$$|T_4| \;\le\; \frac{8L}{\pi}\frac{1}{k_0^4}\int_{\frac{k_0}{2}}^{k_0-\frac{C_0}{4}} \frac{1}{|\zeta-k|}\, d\zeta \le \frac{64L}{C_0\pi}\frac{1}{k_0^4}\left(\frac{k_0}{2} - \frac{C_0}{4}\right).$$

Finally, we can explicitly compute the integral in the fifth and remaining term. Let $\log(z)$ denote the main branch of the logarithm on $\mathbb{C}\backslash\mathbb{R}_{\le 0}$ with $-\pi < \arg(z) < \pi$. Then

$$\int_{k_0-\frac{C_0}{4}}^{k_0} \frac{1}{\zeta-k}\, d\zeta = \log(k_0-k) - \log(k_0-C_0/4-k).$$

In addition, we have $C_0/8 \le |k_0 - C_0/4 - k| \le 3C_0/8$, so

$$|T_5| \;\le\; \frac{L}{2\pi}\frac{1}{k_0^4}\left(|\log(|k-k_0|)| + \max_{C_0/8\le s\le 3C_0/8}|\log(s)| + 2\pi\right).$$

Adding up all these estimates gives the second statement in the lemma. □

5.2. The Proof of the main result

In this section, we finally combine all the previous results to compute the long-time asymptotics in the similarity region.

First of all, we use the matrix $\hat{v}$ from chapter 3 and cut it off smoothly near the critical points $\pm k_0$. If we consider not the vector, but the matrix Riemann–Hilbert problem corresponding to the cut-off jump matrix, chapter 4 provides an asymptotic expansion for the solution. This is formalized in the next lemma.

LEMMA 5.3. *Assume a similarity region S_C and suppose $\rho > 0$ is small. Let χ be a radially symmetric function in $\mathcal{C}_0^\infty(\mathbb{R}^2)$ with $0 \le \chi \le 1$, $\chi(k) = 0$ for $|k| \ge 2\rho$ and $\chi(k) = 1$ for $|k| \le \rho$. For $(x,t) \in S_C$, consider the Riemann–Hilbert problem*

$$\begin{aligned} M^1_+(k) &= M^1_-(k) V_j(k), && k \in k_0 + \Sigma_j, \quad j = 1,2,3,4, \\ M^1(k) &\to \mathbb{I}, && |k| \to \infty, \end{aligned} \tag{5.2}$$

where

$$V_j(k) = \begin{cases} \chi(k-k_0)\hat{v}(k) + (1-\chi(k-k_0))\mathbb{I}, & k \in k_0 + \Sigma_j,\ |k-k_0| \le 2\rho, \\ \mathbb{I}, & k \in k_0 + \Sigma_j,\ |k-k_0| \ge 2\rho, \end{cases}$$

for $j = 1, ..., 4$. A function $M^1 \colon \mathbb{C}\backslash(k_0 + \Sigma) \to \mathbb{C}^{2\times 2}$ is called a solution for (5.2), if it satisfies the four conditions that are obtained from the conditions in (4.1) by replacing the origin with k_0.
Then there is some $T > 0$, such that for every pair $(x,t) \in S_C$ with $t \ge T$, the Riemann–Hilbert problem (5.2) is uniquely solvable and the solution $M^1(k)$ can be represented as

$$M^1(k) = \mathbb{I} + \frac{1}{\sqrt{48k_0}(k-k_0)} \frac{\mathrm{i}}{t^{1/2}} \begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix} + \frac{1}{k-k_0} E^1(k) + H^1(k),$$

where

$$\nu = -\frac{1}{2\pi}\log(|T(k_0)|^2), \qquad r = R(k_0) \prod_{j=1}^N \frac{k_0 - \mathrm{i}\kappa_j}{k_0 + \mathrm{i}\kappa_k}\, \mathrm{e}^{-2\psi(k_0,k_0)} \mathrm{e}^{2\mathrm{i}\nu\log(2k_0\sqrt{48k_0})},$$

$$\beta = \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(r) + \arg(\Gamma(\mathrm{i}\nu)))} \mathrm{e}^{-t\Phi(k_0)} t^{-\mathrm{i}\nu}$$

$$= \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4 - \arg(R(k_0)) + \arg(\Gamma(\mathrm{i}\nu)))} \prod_{j=1}^N \frac{(k_0 + \mathrm{i}\kappa_j)^2}{(k_0 - \mathrm{i}\kappa_j)^2}\, \mathrm{e}^{2\psi(k_0,k_0)} (192k_0^3)^{-\mathrm{i}\nu} \mathrm{e}^{-t\Phi(k_0)} t^{-\mathrm{i}\nu},$$

and the error terms are of the following order:

$$E^1(k) = O\left(k_0^{-1}t^{-\alpha}\right) \text{ for } 1/2 < \alpha < 1 \quad \text{and} \quad H^1(k) = O\left(k_0^{-1}t^{-1}(k-k_0)^{-2}\right),$$

with estimates holding uniformly for $(x,t) \in S_C$ with $t \ge T$ and $k \in \mathbb{C}\backslash(k_0 + \Sigma)$.
The analogous statement holds true for $-k_0$ in place of k_0, where the representation now takes the form

$$M^2(k) = \mathbb{I} - \frac{1}{\sqrt{48k_0}(k+k_0)} \frac{\mathrm{i}}{t^{1/2}} \begin{pmatrix} 0 & \overline{\beta} \\ -\beta & 0 \end{pmatrix} + \frac{1}{k+k_0} E^2(k) + H^2(k)$$

and the error terms are of the following order:

$$E^2(k) = O\left(k_0^{-1}t^{-\alpha}\right) \text{ for } 1/2 < \alpha < 1 \quad \text{and} \quad H^2(k) = O\left(k_0^{-1}t^{-1}(k+k_0)^{-2}\right),$$

with estimates holding uniformly for $(x,t) \in S_C$ *with* $t \geq T$ *and* $k \in \mathbb{C} \backslash (-k_0 + \Sigma)$.

PROOF. The Riemann–Hilbert problem (5.2) is equivalent to a problem as in (4.1) via the coordinate transformation

$$\zeta(k) = \sqrt{48k_0}(k - k_0), \qquad k(\zeta) = k_0 + \frac{\zeta}{\sqrt{48k_0}}.$$

To be precise, a simple calculation shows that the jump matrices $v_j(\zeta) = V_j(k(\zeta))$, $j = 1, ..., 4$ are of the form supposed in chapter 4. Namely, the corresponding problem is just the one for the phase

$$\Theta(\zeta) = \Phi(k_0, x, t) + \frac{\mathrm{i}}{2}\zeta^2 + 8(48k_0)^{-3/2}\mathrm{i}\zeta^3,$$

the parameter ν defined as above, the coefficient functions

$$R_1(\zeta) = \chi(k(\zeta) - k_0)R(-k(\zeta))\ (2\sqrt{48k_0}k_0 + \zeta)^{-2\mathrm{i}\nu} \prod_{j=1}^{N} \frac{(k(\zeta) + \mathrm{i}\kappa_j)^2}{(k(\zeta) - \mathrm{i}\kappa_j)^2} \mathrm{e}^{2\psi(k(\zeta),k_0)},$$

$$R_2(\zeta) = \chi(k(\zeta) - k_0)R(k(\zeta))\ (2\sqrt{48k_0}k_0 + \zeta)^{2\mathrm{i}\nu} \prod_{j=1}^{N} \frac{(k(\zeta) - \mathrm{i}\kappa_j)^2}{(k(\zeta) + \mathrm{i}\kappa_j)^2} \mathrm{e}^{-2\psi(k(\zeta),k_0)},$$

$$R_3(\zeta) = \chi(k(\zeta) - k_0)\frac{R(-k(\zeta))\ (2\sqrt{48k_0}k_0 + \zeta)^{-2\mathrm{i}\nu}}{1 - R(k(\zeta))R(-k(\zeta))} \prod_{j=1}^{N} \frac{(k(\zeta) + \mathrm{i}\kappa_j)^2}{(k(\zeta) - \mathrm{i}\kappa_j)^2} \mathrm{e}^{2\psi(k(\zeta),k_0)},$$

$$R_4(\zeta) = \chi(k(\zeta) - k_0)\frac{R(k(\zeta))\ (2\sqrt{48k_0}k_0 + \zeta)^{2\mathrm{i}\nu}}{1 - R(k(\zeta))R(-k(\zeta))} \prod_{j=1}^{N} \frac{(k(\zeta) - \mathrm{i}\kappa_j)^2}{(k(\zeta) + \mathrm{i}\kappa_j)^2} \mathrm{e}^{-2\psi(k(\zeta),k_0)},$$

and the additional positive parameter equal to t. It is obviously true that $M^1(k)$ solves (5.2) if and only if $m^1(\zeta) := M^1(k(\zeta))$ solves (4.1) with the jump matrices $v_j(\zeta)$. We will now show that the family $((\Theta, \nu, R_j;\ j = 1, ..., 4)_{(x,t)})_{(x,t)\in S_C}$ satisfies the assumptions of Theorem 4.1. Using Lemma 3.4 and the definition of the coefficient functions, we find that there is a uniform constant $K > 0$ such that

$$\left|R_j'(\zeta)\right| \leq \frac{K}{\sqrt{k_0}} + \frac{K}{(k_0)^{9/2}} \left|\log(|\zeta|)\right|$$

for every $\zeta \in \Sigma$, $(x,t) \in S_C$ and $j = 1, ..., 4$. The factor $R'(k(\zeta))$ stays bounded due to Cauchy's inequality and our assumptions on $R(k)$. It is then easily verified that the conditions for the theorem hold true with

$$\rho_{(x,t)} = 2\rho\sqrt{48k_0}, \qquad L_{(x,t)} = \frac{K}{\sqrt{k_0}}, \qquad L'_{(x,t)} = \frac{K}{(k_0)^{9/2}},$$

$$C_{(x,t)} = 8(48k_0)^{-3/2}, \quad C'_{(x,t)} = 24(48k_0)^{-3/2},$$

and $r_{(x,t)} \in \mathbb{D}$ defined by the above formula. The statements thus follows by simply applying the theorem and reversing the coordinate transform. Within the notation used in Chapter 4, the error terms $E^1(k)$ and $H^1(k)$ are given by

$$E^1(k) = \frac{1}{\sqrt{48k_0}}\ e(\sqrt{48k_0}(k - k_0)) \quad \text{and} \quad H^1(k) = h(\sqrt{48k_0}(k - k_0)).$$

Clearly, $H^1(k)$ is of the claimed order. Since $|r_{(x,t)}| = |R(k_0)|$ by Lemma 3.4 and $|R(s)| = O(s^{-4})$ as $|s| \to \infty$, $s \in \mathbb{R}$ by the results in Section 2.1, we have $(L_{(x,t)} + L'_{(x,t)} + |r_{(x,t)}|) = O(k_0^{-1/2})$ for $(x,t) \in S_C$. But this already implies the

result for $E^1(k)$.
To prove the statement for $-k_0$, fix $(x,t) \in S_C$. By the definition of $\hat{v}$ we have

$$\hat{v}(-k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{v}(k)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for every $k \in \hat{\Sigma}(x,t)$, which can be combined with the radial symmetry of χ to get

$$V^2(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V^1(-k)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad k \in -k_0 + \Sigma.$$

Now we can use this to show that a matrix valued function $M^1(k)$ solves the Riemann–Hilbert problem (5.2) if and only if $M^2(k)$ defined by

$$M^2(k) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M^1(-k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

solves the analogous problem for $-k_0$. Indeed, assume $M^1(k)$ solves (5.2). Then

$$\begin{aligned} M^2_+(k) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M^1_-(-k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M^1_+(-k) V^1(-k)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M^1_+(-k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V^2(k) = M^2_-(k) V^2(k) \end{aligned}$$

for $k \in -k_0 + \Sigma$, which means $M^2(k)$ satisfies the jump condition in the Riemann–Hilbert problem at $-k_0$. Obviously, the holomorphicity on $\mathbb{C}\backslash(-k_0+\Sigma)$, the boundedness at $-k_0$ and the limit relation $M^2(k) = \mathbb{I} + o(1)$ for $|k| \to \infty$ are consequences of the respective properties for $M^1(k)$. The other implication in the equivalence can be obtained similarly. Thus, we may conclude the statement for $-k_0$ from the one for k_0. □

This enables us to deduce an asymptotic expansion for $\hat{m}(k)$ in form of the next theorem.

THEOREM 5.4. *Assume a similarity region S_C and let $(x,t) \in S_C$ with t large. Then $\hat{m}(k)$ can be represented as*

$$\hat{m}(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{t^{1/2}} \frac{\mathrm{i}}{\sqrt{48 k_0}} \left(\frac{1}{k - k_0} \begin{pmatrix} \overline{\beta} & -\beta \end{pmatrix} - \frac{1}{k + k_0} \begin{pmatrix} -\beta & \overline{\beta} \end{pmatrix} \right) + er(k),$$

where for every $1/2 < \alpha < 1$ the error term $er(k)$ is of order $t^{-\alpha}$ uniformly for $(x,t) \in S_C$ and $k \in \mathbb{C}$ with $d(k, \Sigma(\hat{x,t})) \geq \epsilon$.
More precisely, there exist positive constants $C_{\epsilon,\alpha}$ for $\epsilon > 0$, $1/2 < \alpha < 1$ and a uniform positive constant T such that whenever $(x,t) \in S_C$ with $t \geq T$, $\hat{m}(k)$ has the above representation and the error term $er(k,x,t)$ can be estimated by

$$\|er(k,x,t)\| \;\leq\; C_{\epsilon,\alpha}\, t^{-\alpha}$$

for every $k \in \mathbb{C}$ with $d(k, \hat{\Sigma}(x,t)) \geq \epsilon$.

PROOF. First of all, it should be remarked that whenever we claim a term to be of a certain order in this proof, we implicitly mean that the related estimated holds with a constant independent of $(x,t) \in S_C$.
We start by fixing $\epsilon > 0$ and $1/2 < \alpha < 1$. Without loss of generality, we assume that ϵ is so small, that Lemma 5.3 applies for $\rho = \epsilon$. Now, suppose $(x,t) \in S_C$ with

t large. Then we write $D^1 = D(k_0, \epsilon/2)$, $D^2 = D(-k_0, \epsilon/2)$, denote by $M^1(k)$ and $M^2(k)$ the matrix-valued solution from Lemma 5.3 with $\rho = \epsilon$ and let

$$\tilde{M}^1 := \frac{\mathrm{i}}{\sqrt{48}} \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} \qquad \text{and} \qquad \tilde{M}^2 := \frac{\mathrm{i}}{\sqrt{48}} \begin{pmatrix} 0 & -\overline{\beta} \\ \beta & 0 \end{pmatrix}.$$

We try to move the jump contour away from the points $\pm k_0$ and redefine $\hat{m}(k)$ by

$$\breve{m}(k) := \begin{cases} \hat{m}(k) M^1(k)^{-1}, & k \in D^1, \\ \hat{m}(k) M^2(k)^{-1}, & k \in D^2, \\ \hat{m}(k), & \text{else}, \end{cases}$$

for $k \in \mathbb{C} \backslash (\hat{\Sigma}(x,t) \cup \{|k - k_0| = \epsilon/2 \text{ or } |k + k_0| = \epsilon/2\})$. Then $\breve{m}(k)$ is continuous on the parts of $\hat{\Sigma}$ that are closer than $\epsilon/2$ to $\pm k_0$, and bounded near the points $\pm k_0$. By the theorems of Morera and Riemann, it can be extended holomorphically to the discs D^1 and D^2. So, $\breve{m}(k)$ is discontinuous only along the contour $\breve{\Sigma}(x,t)$ consisting of the remaining parts of $\hat{\Sigma}(x,t)$ and the circles around the points $\pm k_0$. If we orient these circles counterclockwise, the jump matrix relating the continuous limits from the left and right side is given by

$$\breve{v}(k) = \begin{cases} M^1(k)^{-1}, & k \in \partial D^1, \\ M^2(k)^{-1}, & k \in \partial D^2, \\ \hat{v}(k), & \text{else}. \end{cases}$$

Now we conclude from Lemma 5.3, that $\sup_{k \in \partial D^1} \|\mathbb{I} - M^1(k)\| \leq 1/2$ for $(x,t) \in S_C$ with t sufficiently large and that in this case

$$(5.3) \qquad M^1(k)^{-1} = \mathbb{I} + (\mathbb{I} - M^1(k)) + (\mathbb{I} - M^1(k))^2 \sum_{n=0}^{\infty} (\mathbb{I} - M^1(k))^n =$$
$$= \mathbb{I} - \frac{\tilde{M}^1}{\sqrt{k_0} t^{1/2} (k - k_0)} + O(k_0^{-1} t^{-\alpha})$$

with the error term uniform for $k \in \partial D^1$. By the results in Section 2.1, we have $\beta = O(1/{k_0}^4)$. Together with Lemma 5.1, this implies $\breve{w}(x,t) = \breve{v}(x,t) - \mathbb{I}$ has order

$$(5.4) \qquad \|\breve{w}(x,t)\|_{L^\infty(\breve{\Sigma}(x,t))} = O(k_0^{-1} t^{-1/2}), \qquad \|\breve{w}(x,t)\|_{L^2(\breve{\Sigma}(x,t))} = O(k_0^{-1} t^{-1/2}).$$

Next, we represent $\breve{m}(k)$ as a Cauchy integral and write

$$\breve{m}(k) - \begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2\pi\mathrm{i}} \int_{\breve{\Sigma}} \frac{\breve{m}_+(s) - \breve{m}_-(s)}{s - k} \, ds = \frac{1}{2\pi\mathrm{i}} \int_{\breve{\Sigma}} \frac{\breve{m}_-(s) \breve{w}(s)}{s - k} \, ds,$$

for $k \in \mathbb{C} \backslash \breve{\Sigma}(x,t)$. In fact, this step can be justified by Theorem A.5 after extending $\breve{\Sigma}(x,t)$ to a Carleson jump contour $\Gamma(x,t)$ and using (3.12) plus Lemma A.6 to show that $\breve{m}(k) \in \dot{E}^2(\hat{\mathbb{C}} \backslash \Gamma(x,t))$. This formula allows us to use the connection between Riemann–Hilbert problems and singular integral equations to estimate the L^2 norm of $\breve{m}$. Since $\breve{\Sigma}(x,t)$ can be viewed as a part of a Carleson jump contour $\Gamma(x,t)$, the statements in Theorem A.4 are valid for $\breve{\Sigma}(x,t)$ as well. Denote by $\mathcal{C}_\pm$ the respective Cauchy operators on $L^2(\breve{\Sigma}(x,t))$ and expand them to $L^2(\breve{\Sigma}(x,t))^2$ by applying $\mathcal{C}_\pm$ componentwise. Then

$$\mathcal{C}_{\breve{w}} f := \mathcal{C}_-(f \breve{w})$$

is a bounded, linear operator on $L^2(\breve{\Sigma}(x,t))^2$. The properties of $\mathcal{C}_\pm$ imply

$$\breve{m}_-\breve{w} + \breve{m}_- = \breve{m}_+ = \begin{pmatrix}1 & 1\end{pmatrix} + \mathcal{C}_+(\breve{m}_-\breve{w}) = \begin{pmatrix}1 & 1\end{pmatrix} + \breve{m}_-\breve{w} + \mathcal{C}_-(\breve{m}_-\breve{w})$$

almost everywhere on $\breve{\Sigma}(x,t)$. In particular, we have the singular integral equation

$$(\mathrm{id} - \mathcal{C}_{\breve{w}})\left(\breve{m}_- - \begin{pmatrix}1 & 1\end{pmatrix}\right) = \mathcal{C}_-\left(\begin{pmatrix}1 & 1\end{pmatrix}\cdot\breve{w}\right) \qquad \text{in } L^2(\breve{\Sigma}(x,t))^2.$$

On the other hand, a translation argument shows that

$$\|\mathcal{C}_-\|_{L^2(\breve{\Sigma}(x,t))\to L^2(\breve{\Sigma}(x,t))} = O(1).$$

In view of (5.4), this means there is a $T'' > 0$ such that the operator norm of $\mathcal{C}_{\breve{w}}$ is smaller than $1/2$ for every $(x,t) \in S_C$ with $t \geq T''$. Assuming this, a von Neumann argument gives that $\mathbb{I} - \mathcal{C}_{\breve{w}(x,t)}$ is invertible and that

$$\left\|\breve{m}_- - \begin{pmatrix}1 & 1\end{pmatrix}\right\|_{L^2(\breve{\Sigma}(x,t))} \leq 2\left\|\mathcal{C}_-\left(\begin{pmatrix}1 & 1\end{pmatrix}\cdot\breve{w}(x,t)\right)\right\|_{L^2(\breve{\Sigma}(x,t))} = O(t^{-1/2}). \tag{5.5}$$

Finally, assume $k \in \mathbb{C}$ with $d(k, \hat{\Sigma}(x,t)) \geq \epsilon$. Then Lemma 5.1 and the above imply

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{\breve{\Sigma}\cap\hat{\Sigma}} \frac{\breve{m}_-(s)\breve{w}(s)}{s-k}\,ds &= O\left(\left\|\breve{m}_- - \begin{pmatrix}1 & 1\end{pmatrix}\right\|_{L^2(\hat{\Sigma}\cap\breve{\Sigma})}\|\hat{w}\|_{L^2(\hat{\Sigma}\cap\breve{\Sigma})} + \|\hat{w}\|_{L^1(\hat{\Sigma}\cap\breve{\Sigma})}\right)\\ &= O\left(k_0^{-1}t^{-1}\right)\end{aligned}$$

and therefore

$$\begin{aligned}\hat{m}(k) - \begin{pmatrix}1 & 1\end{pmatrix} &= \breve{m}(k) - \begin{pmatrix}1 & 1\end{pmatrix}\\ &= \sum_{j=1}^{2}\frac{1}{2\pi\mathrm{i}}\int_{\partial D^j}\frac{\breve{m}_-(s)\breve{w}(s)}{s-k}\,ds + O\left(k_0^{-1}t^{-1}\right)\\ &= \sum_{j=1}^{2}\frac{1}{2\pi\mathrm{i}}\int_{\partial D^j}\begin{pmatrix}1 & 1\end{pmatrix}\frac{M^j(s)^{-1} - \mathbb{I}}{s-k}\,ds + O\left(k_0^{-1}t^{-1}\right)\\ &= \sum_{j=1}^{2}\frac{\begin{pmatrix}1 & 1\end{pmatrix}\tilde{M}^j}{-\sqrt{k_0}t^{1/2}}\frac{1}{2\pi\mathrm{i}}\int_{\partial D^j}\frac{1}{(s-k)(s+(-1)^jk_0)}\,ds + O\left(k_0^{-1}t^{-\alpha}\right)\\ &= \frac{1}{\sqrt{k_0}t^{1/2}}\frac{1}{k-k_0}\begin{pmatrix}1 & 1\end{pmatrix}\tilde{M}^1 + \frac{1}{\sqrt{k_0}t^{1/2}}\frac{1}{k+k_0}\begin{pmatrix}1 & 1\end{pmatrix}\tilde{M}^2 + O\left(k_0^{-1}t^{-\alpha}\right)\\ &= \frac{1}{t^{1/2}}\frac{1}{k-k_0}\frac{\mathrm{i}}{\sqrt{48k_0}}\begin{pmatrix}\overline{\beta} & -\beta\end{pmatrix} - \frac{1}{t^{1/2}}\frac{1}{k+k_0}\frac{\mathrm{i}}{\sqrt{48k_0}}\begin{pmatrix}-\beta & \overline{\beta}\end{pmatrix} + O\left(k_0^{-1}t^{-\alpha}\right),\end{aligned}$$

with all estimates independent of k. The theorem is proven. □

To reconstruct the solution $q(x,t)$ of the Korteweg–de Vries equation, we will not need the asymptotics for the solution $\hat{m}(k)$ on the whole complex plane. In fact, it will suffice to know them on the imaginary axis and away from the origin. We thus define

$$\mathcal{I} = \{k \in \mathrm{i}\mathbb{R};\ |k| \geq 2(\kappa_N + \check{\epsilon})\}$$

and investigate the error term $er(k)$ for $k \in \mathcal{I}$ more closely.

Lemma 5.5. *Assume a similarity region S_C. Then there exists some $T > 0$, such that for $(x,t) \in S_C$ with $t \geq T$, the error term $er(k,x,t)$ on $\mathcal{I}$ can be written as*

$$er(k,x,t) = \frac{1}{k}a(x,t) + \frac{1}{k^2}\widetilde{er}(k,x,t), \qquad k \in \mathcal{I}, \tag{5.6}$$

where the coefficient $a(x,t) \in \mathbb{C}^2$ *has order*

$$a(x,t) = O(t^{-\alpha}) \text{ for every } 1/2 < \alpha < 1,$$

with estimates holding uniformly for $(x,t) \in S_C$ *with* $t \geq T$, *and the error term* $\widetilde{er}(k,x,t)$ *has order*

$$\widetilde{er}(k,x,t) = O(t^{-\alpha}) \text{ for every } 1/2 < \alpha < 1,$$

with estimates holding uniformly for $(x,t) \in S_C$ *with* $t \geq T$ *and* $k \in \mathcal{I}$.

Proof. First of all, we fix some small $\epsilon > 0$ and define $D^{1,2}$, $\check{\Sigma}(x,t)$, $\check{m}(x,t)$, $M^{1,2}$, $\tilde{M}^{1,2}$ and $\check{w}(x,t)$ with respect to this ϵ as in the proof of Theorem 5.4. For every $k \in \mathbb{C}$ with $d(k, \hat{\Sigma}(x,t)) \geq \epsilon$, the argument there yields

$$er(k,x,t) = \frac{1}{2\pi \mathrm{i}} \int_{\check{\Sigma}} \frac{f(s)}{s-k}\, ds,$$

where f is given by

$$f(s) := \check{m}_-(s)\check{w}(s), \qquad \text{for } s \in \check{\Sigma}(x,t) \cap \hat{\Sigma}(x,t),$$

and

$$f(s) := \big(\check{m}_-(s) - \begin{pmatrix} 1 & 1 \end{pmatrix}\big) \cdot \check{w}(s) + \begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \left(\check{w}(s) + \frac{\tilde{M}^j}{(k_0 t)^{1/2}(s + (-1)^j k_0)} \right),$$

for $s \in \partial D^j$, $j = 1,2$. In particular, this holds true for $k \in \mathcal{I}$. Next, we notice that the factor $1/(s-k)$ can be rewritten as

$$\frac{1}{s-k} = -\frac{1}{k} + \frac{1}{k^2} \frac{s}{\frac{s}{k} - 1}, \quad s \neq k.$$

But this means the desired representation (5.6) holds, if we put

$$a(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{\check{\Sigma}} -f(s)\, ds \quad \text{and} \quad \widetilde{er}(k,x,t) = \frac{1}{2\pi \mathrm{i}} \int_{\check{\Sigma}} \frac{s}{\frac{s}{k} - 1} f(s)\, ds$$

for $(x,t) \in S_C$ and $k \in \mathcal{I}$. More precisely, $a(x,t)$ is given by

$$\begin{aligned} a(x,t) = &-\sum_{j=1}^{2} \frac{1}{2\pi \mathrm{i}} \int_{\partial D^j} \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\check{w}(s) + \frac{\tilde{M}^j}{(k_0 t)^{1/2}(s + (-1)^j k_0)} \right) ds \\ &-\sum_{j=1}^{2} \frac{1}{2\pi \mathrm{i}} \int_{\partial D^j} \big(\check{m}_-(s) - \begin{pmatrix} 1 & 1 \end{pmatrix}\big)\, \check{w}(s)\, ds - \frac{1}{2\pi \mathrm{i}} \int_{\check{\Sigma} \cap \hat{\Sigma}} \check{m}_-(s)\check{w}(s)\, ds. \end{aligned}$$

By the estimates (5.3), (5.5), (5.4) and Lemma 5.1, $a(x,t)$ is of order $k_0^{-1} t^{-\alpha}$ for every $1/2 < \alpha < 1$, where the corresponding estimate is uniform with respect to $(x,t) \in S_C$ with t large.
Turning to estimating $\widetilde{er}(k,x,t)$, we find there is some uniform constant $K > 0$, such that

$$\frac{1}{\left|\frac{s}{k} - 1\right|} \leq K$$

for every $s \in \mathbb{C}$ in the strip $-\kappa_N - \check{\epsilon} \leq \mathrm{Im}(s) \leq \kappa_N + \check{\epsilon}$ and every $k \in \mathcal{I}$. Indeed, assume first that s is contained in the square S centered at the origin with side length $2(\kappa_N + \check{\epsilon})$. Then the above estimate follows from a compactness argument. On the other hand, if s is in the strip, but not in the square, s/k must be contained

in $A = \{re^{i\phi};\ r \geq 0,\ \phi \in [-3\pi/4, -\pi/4] \cup [\pi/4, 3\pi/4]\}$. But this means that $|s/k - 1| \geq d(1, A) > 0$.
Now assume $(x,t) \in S_C$ with t large. For $s \in \partial D^1$ or $s \in \partial D^2$, we have the estimate $|s(s/k-1)^{-1}| \leq K(1+\epsilon/C_0)k_0$. Using again (5.3), (5.5) and (5.4) leads to

$$\sum_{j=1}^{2} \frac{1}{2\pi i} \int_{\partial D^j} \frac{s \begin{pmatrix} 1 & 1 \end{pmatrix}}{\frac{s}{k} - 1} \left(\breve{w}(s) + \frac{\tilde{M}^j}{(k_0 t)^{1/2}(s + (-1)^j k_0)} \right) ds = O(t^{-\alpha}),$$

$$\sum_{j=1}^{2} \frac{1}{2\pi i} \int_{\partial D^j} \frac{s}{\frac{s}{k} - 1} \left(\breve{m}_-(s) - \begin{pmatrix} 1 & 1 \end{pmatrix} \right) \breve{w}(s)\, ds = O(t^{-\alpha}),$$

for every $1/2 < \alpha < 1$, with estimates independent of (x,t) and $k \in \mathcal{I}$. To finish the proof, we thus only have to show

$$\frac{1}{2\pi i} \int_{\breve{\Sigma} \cap \hat{\Sigma}} \frac{s}{\frac{s}{k} - 1}\, \breve{m}_-(s) \breve{w}(s)\, ds = O(t^{-\alpha}),$$

for every $1/2 < \alpha < 1$, where the estimate is uniform as above. Denote by $L(x,t)$ the contour consisting of the four infinite, straight parts of $\hat{\Sigma}(x,t)$. An inspection of $\hat{\Sigma}(x,t)$ shows that there is a uniform constant $K' > 0$ with $|s| \leq K' k_0$ for every $s \in (\hat{\Sigma} \backslash L)(x,t)$ and $(x,t) \in S_C$. Therefore,

$$\frac{1}{2\pi i} \int_{(\breve{\Sigma} \cap \hat{\Sigma}) \backslash L} \frac{s}{\frac{s}{k} - 1} \breve{m}_-(s) \breve{w}(s)\, ds = O(t^{-1})$$

by Lemma 5.1 and (5.5). A straightforward estimate then gives

$$\| s \breve{w}(s) \|_{L^1(L(x,t))} + \| s \breve{w}(s) \|_{L^2(L(x,t))} = O(t^{-1}),$$

which proves that the integral over $L(x,t)$ is of the correct order as well. □

Finally, the asymptotic connection between the Riemann–Hilbert problem and the potential can be used to obtain the desired result.

THEOREM 5.6. *Consider the similarity region S_C for some $C > 0$. Then for $(x,t) \in S_C$ with t sufficiently large,*

$$\begin{aligned} \int_x^\infty q(y,t)\, dy = &- 4 \sum_{j=1}^{N} \kappa_j - \frac{1}{\pi} \int_{-k_0}^{k_0} \log(|T(\zeta)|^2)\, d\zeta \\ &- \sqrt{\frac{\nu(k_0)}{3 k_0 t}} \cos\left(16 t k_0^3 - \nu(k_0) \log(192 t k_0^3) + \delta(k_0)\right) + er_Q(x,t) \end{aligned}$$

respectively

$$q(x,t) = \sqrt{\frac{4 \nu(k_0) k_0}{3t}} \sin\left(16 t k_0^3 - \nu(k_0) \log(192 t k_0^3) + \delta(k_0)\right) + er_q(x,t),$$

where

$$|er_Q(x,t)| \leq C_\alpha t^{-\alpha} \qquad \text{and} \qquad |er_q(x,t)| \leq C_\alpha t^{-\alpha}$$

for every $1/2 < \alpha < 1$. Here, $k_0 = \sqrt{-\frac{x}{12t}}$ and

$$\nu(k_0) = -\frac{1}{2\pi} \log(|T(k_0)|^2),$$

$$\delta(k_0) = \frac{\pi}{4} - \arg(R(k_0)) + \arg(\Gamma(\mathrm{i}\nu(k_0))) + 4\sum_{j=1}^{N} \arctan\left(\frac{\kappa_j}{k_0}\right) - 2\mathrm{i}\psi(k_0, k_0),$$

with $\psi(k_0, k_0)$ defined as in Lemma 3.4.

Proof. Fix $(x,t) \in S_C$ with t large. By Lemma 2.7,

$$1 - \frac{1}{2\mathrm{i}k} Q(x,t) = \hat{m}_1(k,x,t) T(k,k_0) + O\left(\frac{1}{k^2}\right)$$

for $k \in \mathcal{I}$, $|k| \to \infty$. Thus, (3.6) and Theorem 5.4 imply

$$Q(x,t) = 2T_1(k_0) + \frac{2}{t^{1/2}\sqrt{48k_0}}\left(\overline{\beta}\frac{k}{k-k_0} + \beta\frac{k}{k+k_0}\right) T(k,k_0) - 2\mathrm{i}k\, er_1(x,t,k)\, T(k,k_0) + O\left(\frac{1}{k}\right).$$

If we let $|k|$ tend to infinity for $k \in \mathcal{I}$, the first formula follows from Lemma 5.5. Now investigating $q(x,t)$, we again use Lemma 2.7 to get

$$\begin{aligned} 1+\frac{q(x,t)}{2k^2} &= m_1(k,x,t)m_2(k,x,t) + o\left(\frac{1}{k^2}\right) = \hat{m}_1(k,x,t)\hat{m}_2(k,x,t) + o\left(\frac{1}{k^2}\right) \\ &= 1 + (b_1(k) + b_2(k)) + (er_1(k) + er_2(k)) \\ &\quad + b_1(k)b_2(k) + b_1(k)er_2(k) + b_2(k)er_1(k) + er_1(k)er_2(k) + o\left(\frac{1}{k^2}\right) \end{aligned}$$

for $k \in \mathcal{I}$, $|k| \to \infty$, where we have put

$$b(k,x,t) = \frac{1}{t^{1/2}}\frac{\mathrm{i}}{\sqrt{48k_0}}\left(\frac{1}{k-k_0}\begin{pmatrix}\overline{\beta} & -\beta\end{pmatrix} - \frac{1}{k+k_0}\begin{pmatrix}-\beta & \overline{\beta}\end{pmatrix}\right).$$

As before, we want to take limits and find

$$\lim_{|k|\to\infty,\ k\in\mathcal{I}} 2k^2(b_1(k,x,t) + b_2(k,x,t)) = \sqrt{\frac{4k_0}{3t}}\,\mathrm{Im}(\beta).$$

A straightforward calculation using Lemma 5.5 shows that

$$\lim_{|k|\to\infty,\ k\in\mathcal{I}} 2k^2\left(b_1(k)b_2(k) + b_1(k)er_2(k) + b_2(k)er_1(k) + er_1(k)er_2(k)\right)$$

exists and is of order t^{-1} uniformly with respect to (x,t). Since $\hat{m}(k,x,t)$ is symmetric, $er(k,x,t)$ is symmetric as well and therefore

$$\begin{aligned} 2k^2(er_1(k,x,t) + er_2(k,x,t)) &= 2k^2(er_1(k,x,t) + er_1(-k,x,t)) \\ &= 2(\widetilde{er}_1(k,x,t) + \widetilde{er}_1(-k,x,t)). \end{aligned}$$

We can thus conclude the second statement from Lemma 5.5 by taking the limit $|k| \to \infty$, $k \in \mathcal{I}$. □

APPENDIX A

Scalar Riemann–Hilbert problems and the Cauchy operator

In this appendix, a few basic results from the theory of scalar Riemann–Hilbert problems will be prepared for reference in the rest of the text. Roughly speaking, the additive Riemann–Hilbert problem for a complex-valued function $\phi(z)$ defined on a contour L in the complex plane consists of finding an analytic function $m(z)$ on $\mathbb{C}\backslash L$, such that in some sense there exist limits $m_\pm(z)$ of $m(z)$ for z converging from the left resp. right side to L and

$$m_+(z) - m_-(z) = \phi(z), \qquad z \in L. \tag{A.1}$$

There are several ways to formalize the way in which the limits of $m(z)$ to the contour are taken, for example the existence of the limit in the L^2 norm or in the sense of a continuous extension. Which precise formulation of the problem is reasonable depends on the properties of $\phi(z)$ and the contour L. In many situations, a solution is given in form of the Cauchy integral

$$\mathcal{C}\phi\ (z) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\phi(s)}{s-z}\, ds, \qquad z \in \mathbb{C}\backslash L.$$

A.1. The Riemann–Hilbert problem for Hölder continuous functions

Within classical approaches, one usually supposes the right hand side of the Riemann–Hilbert problem to satisfy some Hölder condition on the jump contour. The standard reference for this theory is [**18**], and the following definitions and results can be found there.

A smooth arc L is a curve in the complex plane that can be represented by a function $\gamma \in \mathcal{C}^1([0,1])$ with $\gamma'(s) \neq 0$ for all $s \in [0,1]$ and $\gamma(s) \neq \gamma(t)$ for $s,t \in [0,1]$, $s \neq t$. In this case, the points $\gamma(0)$ and $\gamma(1)$ are called the end points of L. Furthermore, we orient L in such a way that $\gamma(s)$ traverses L in the positive direction, if s increases. Whenever we speak of the left or right side of L, we implicitly refer to this fixed orientation.

Suppose now $\Phi(z)$ is a continuous function defined on $U\backslash L$, where U is a neighborhood of a smooth arc L. Let s be a fixed point on L not equal to an endpoint. Then $\Phi(z)$ is said to be continuous from the left (resp. right) at s, if there is a limit value $\Phi_+(s)$ (resp. $\Phi_-(s)$) such that $\Phi(z_n)$ tends to $\Phi_+(s)$ (resp. to $\Phi_-(s)$), whenever $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{C}\backslash L$ converging to s with z_n on the left (resp. on the right) side of L for all $n \in \mathbb{N}$. In the case that $\Phi(z)$ is continuous from the left (resp. right) at every point s of L except the endpoints, it is easily shown that the limit function $s \mapsto \Phi_+(s)$ (resp. $s \mapsto \Phi_-(s)$) is a continuous function on L without

the endpoints.
Let t be an endpoint of L. Then $\Phi(z)$ is said to be continuous at t, if there is a limit value $\Phi(t)$ such $\Phi(z_n)$ converges to $\Phi(t)$, whenever $(z_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{C}\backslash L$ converging to t. It is clear that, if $\Phi(z)$ is continuous in the endpoint t and continuous from the left (resp. right) in every point s of L without the endpoints, the limit function $s \mapsto \Phi_+(s)$ (resp. $s \mapsto \Phi_-(s)$) can be extended continuously to t by setting $\Phi_+(t) = \Phi(t)$ (resp. $\Phi_-(t) = \Phi(t)$).

Assume an index $0 < \mu \leq 1$ and a complex-valued function f, defined on a subset $U \subseteq \mathbb{C}$. Then f is said to be Hölder continuous of index μ, if there is a positive constant L such that

$$|f(x) - f(y)| \;\leq\; L|x-y|^\mu$$

for all x, y in U.

Theorem A.1. *Suppose L is a smooth arc and $\phi(s)$ is a function on L, which is Hölder continuous of some index $0 < \mu \leq 1$. Then the Cauchy integral*

$$\mathcal{C}\phi\ (z) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\phi(s)}{s-z}\, ds$$

defines a holomorphic function on $\mathbb{C}\backslash L$. For every point s on L not coinciding with an endpoint, $\mathcal{C}\phi(z)$ is continuous from the left and right in s and the decomposition formula

$$(\mathcal{C}\phi)_+\ (s) - (\mathcal{C}\phi)_-\ (s) = \phi(s)$$

holds. In addition, $\mathcal{C}\phi(z)$ is continuous in every endpoint t of L with $\phi(t) = 0$.

A.2. The Riemann–Hilbert problem for L^2 functions

If the right-hand side $\phi(s)$ for the Riemann–Hilbert problem in (A.1) is not continuous, it seems no longer reasonable to take the limits to L in the sense of a continuous extension. However, the classical theory transfers to the L^2 setting in form of the next theorem.

Theorem A.2. *For $\phi \in L^2(\mathbb{R})$, define the Cauchy operator applied to ϕ via*

$$\mathcal{C}\phi\ (z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \frac{\phi(s)}{s-z}\, ds, \qquad z \in \mathbb{C}\backslash\mathbb{R},$$

and denote its restrictions to the lines $\pm\mathrm{i}\epsilon + \mathbb{R}$ by

$$\mathcal{C}_{\pm,\epsilon}\phi(x) := \mathcal{C}\phi\ (x \pm \mathrm{i}\epsilon), \qquad x \in \mathbb{R}.$$

Then $\mathcal{C}\phi\ (z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and there exist functions $\mathcal{C}_\pm\phi \in L^2(\mathbb{R})$ such that

$$\lim_{\epsilon\searrow 0} \mathcal{C}_{\pm,\epsilon}\phi = \mathcal{C}_\pm\phi \text{ in } L^2(\mathbb{R}) \quad \text{and} \quad \lim_{\epsilon\searrow 0} \mathcal{C}\phi\ (x \pm \mathrm{i}\epsilon) = \mathcal{C}_\pm\phi\ (x) \text{ for a.e. } x \in \mathbb{R}.$$

Furthermore, the mappings $\mathcal{C}_\pm \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $\phi \mapsto \mathcal{C}_\pm\phi$ are bounded linear operators and

$$\mathcal{C}_+\phi - \mathcal{C}_-\phi = \phi$$

for every $\phi \in L^2(\mathbb{R})$.

Proof. The theorem can be obtained from the findings in [**22**] as follows: For $f \in L^2(\mathbb{R})$, we introduce the Poisson integral $\mathcal{P}f(z)$ on the upper half plane by setting

$$\mathcal{P}f(x+\mathrm{i}y) = \int_{\mathbb{R}} (\mathcal{F}f)(s) \mathrm{e}^{2\pi \mathrm{i} sx} \mathrm{e}^{-2\pi|s|y}\, ds, \qquad x \in \mathbb{R},\ y > 0.$$

Here, $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R})$ with norming constants chosen such that

$$\mathcal{F}f(s) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi \mathrm{i} ts}\, dt, \qquad f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

The definition of the Fourier transform given here is not the same as in the proof of Lemma 4.2, but simplifies the below computations. By Theorem 1 in Section III.2 of [**22**], the functions

$$\mathcal{P}_y f(x) := \mathcal{P}f\ (x+\mathrm{i}y), \qquad x \in \mathbb{R},$$

converge to f in the sense that

$$\lim_{y \searrow 0} \mathcal{P}_y f = f \text{ in } L^2(\mathbb{R}) \quad \text{and} \quad \lim_{y\searrow 0} \mathcal{P}_y f\ (x) = f(x) \text{ for a.e. } x \in \mathbb{R}.$$

Note that our definition of the Poisson integral coincides with the one given in formula (26) on page 60 of [**22**], since the author there uses the definition of the Fourier transform with $\hat{f}(s) = \int_{\mathbb{R}} f(t)\mathrm{e}^{2\pi i ts}\, dt$ for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.
Next, we introduce the Hilbert transform on $L^2(\mathbb{R})$ by setting

$$Hf := \mathcal{F}^{-1}(-\mathrm{i}\,\mathrm{sign}(s)\ (\mathcal{F}f)(s)) \in L^2(\mathbb{R})$$

for every $f \in L^2(\mathbb{R})$. Clearly, H is a bounded linear operator on $L^2(\mathbb{R})$.
Assume now, $\phi \in L^2(\mathbb{R})$ is fixed. If we prove that

$$\text{(A.2)} \qquad \mathcal{P}\left(\frac{1}{2}\phi + \frac{\mathrm{i}}{2}H\phi\right)\ (z) = \mathcal{C}\phi\ (z) \qquad z \in \mathbb{C},\ \mathrm{Im}(z) > 0,$$

the first part of the theorem will follow with $\mathcal{C}_+\phi = 1/2\phi + \mathrm{i}/2H\phi$. To this end, fix $y > 0$ and define the Poisson kernel $K_y(s)$ by

$$K_y(s) := \frac{1}{\pi}\frac{y}{s^2+y^2}, \quad s \in \mathbb{R}.$$

As shown for example in [**16**] (see Example 1.5 and Theorem 1.1),

$$(\mathcal{F}K_y)\ (s) = \mathrm{e}^{-2\pi|s|y} \quad \text{for almost every } s \in \mathbb{R},$$

which means we can apply the convolution and the Fourier inversion theorem to get that

$$\mathcal{P}\left(\frac{1}{2}\phi + \frac{\mathrm{i}}{2}H\phi\right)\ (x+\mathrm{i}y) = \frac{1}{2}\int_{\mathbb{R}} K_y(x-s)(\phi(s) + \mathrm{i}H\phi(s))\, ds$$

for almost every $x \in \mathbb{R}$. A simple argument using that $\mathcal{F}$ is unitary on $L^2(\mathbb{R})$ gives that the adjoint operator of the Hilbert transform is $H^* = -H$. In addition, we have (see (2.6) on page 459 of [**15**])

$$H\left(K_x(x-\cdot)\right)\ (s) = \frac{1}{\pi}\ \frac{s-x}{(s-x)^2+y^2} \quad \text{for almost every } s \in \mathbb{R},$$

and therefore

$$\begin{aligned}\mathcal{P}\left(\frac{1}{2}\phi + \frac{\mathrm{i}}{2}H\phi\right)\,(x+\mathrm{i}y) &= \frac{1}{2}\int_{\mathbb{R}} K_y(x-s)\phi(s)\,ds + \frac{\mathrm{i}}{2}(H\phi, K_y(x-\cdot))_{L^2(\mathbb{R})} \\ &= \frac{1}{2}\int_{\mathbb{R}} K_y(x-s)\phi(s)\,ds - \frac{\mathrm{i}}{2}(\phi, H(K_y(x-\cdot))_{L^2(\mathbb{R})} \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} \frac{s-x+\mathrm{i}y}{(s-x)^2+y^2}\phi(s)\,ds \\ &= \mathcal{C}\phi\,(x+\mathrm{i}y)\end{aligned}$$

for almost every $x \in \mathbb{R}$. This proves (A.2), since both sides of the equation are easily seen to be continuous on $\mathrm{i}y + \mathbb{R}$ and $y > 0$ was arbitrary.
Finally, the statement concerning the lower half plane can be inferred from the one for the upper half plane upon noticing that

$$\begin{aligned}\mathcal{C}_{-,\epsilon}\phi\,(x) = \mathcal{C}\phi\,(x-\mathrm{i}\epsilon) &= -\int_{\mathbb{R}} \frac{\phi(-s)}{s-(-x+\mathrm{i}\epsilon)}\,ds \\ &= -(\mathcal{C}\phi(-s))\,(-x+\mathrm{i}\epsilon) = -(\mathcal{C}_{+,\epsilon}\phi(-s))\,(-x).\end{aligned}$$

Applying the result for the upper half plane, a straightforward calculation gives the theorem with $\mathcal{C}_-\phi = -1/2\phi + \mathrm{i}/2H\phi$.

□

One major difference between this result and the one for Hölder continuous functions is the following: In the classical theory, we have the existence of the limit of $\mathcal{C}\phi(z)$, if z converges to a point on L on an arbitrary path that only has to lie to one side of L, whereas in the L^2 case the path has to be perpendicular to the real line. However, the above result can be generalized from paths parallel to the imaginary axis to paths contained in arbitrarily large cones. This will be done next.

Given any $0 < \alpha < \pi/2$, let

$$\Gamma_\alpha = \{z \in \mathbb{C}\backslash\mathbb{R}_{\le 0} :\ -\alpha \le \arg(z) \le \alpha\},$$

where $\arg(z)$ is the branch of the argument function with $-\pi \le \arg(z) < \pi$. For x in $\mathbb{R}$, we define the left and right α-cone at x by setting

$$\Gamma_\alpha^+(x) = x + \mathrm{i}\cdot\Gamma_\alpha \qquad \text{and} \qquad \Gamma_\alpha^-(x) = x + (-\mathrm{i})\cdot\Gamma_\alpha.$$

Assume, f is a complex-valued function on $\mathbb{R}$ and g is a complex-valued function on $\mathbb{C}\backslash\mathbb{R}$. Let $x \in \mathbb{R}$. We say that g converges non-tangentially from the left resp. right to f at x if

$$\lim_{z\to x,\ z\in\Gamma_\alpha^+(x)} g(z) = f(x) \qquad \text{resp.} \qquad \lim_{z\to x,\ z\in\Gamma_\alpha^-(x)} g(z) = f(x)$$

for every $0 < \alpha < \pi/2$.

Theorem A.3. *Suppose $\phi \in L^2(\mathbb{R})$. Then $\mathcal{C}\phi$ converges non-tangentially from the left (resp. right) to $\mathcal{C}_+\phi$ (resp. $\mathcal{C}_-\phi$) at almost every $x \in \mathbb{R}$.*

Proof. Similarly to the previous theorem, this follows from the theory in [**22**] (see Theorem 1 in Section VII.1). □

All in all, this shows that the results for Hölder continuous functions in some sense transfer to the L^2 space on the real line $\mathbb{R}$. In the applications, however, there is also need for analogous results on contours of less regularity. We will now introduce a class of contours, for which the notion of non-tangential convergence is still meaningful, and formulate properties of the Cauchy operator for L^2 functions on these contours. The following definitions are taken from [**13**].

A subset $\Gamma \subseteq \mathbb{C}$ is called an arc, if it contains at least two points and is homeomorphic to a (bounded or unbounded, open, closed or half-open) interval $I \subseteq \mathbb{R}$. Suppose $\phi\colon I \to \Gamma$ is a homeomorphism and set $\operatorname{int}(I) = (a, b)$ with $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{-\infty\}$. If $\lim_{t \searrow a} \phi(t)$ or $\lim_{t \nearrow b} \phi(t)$ exists and is finite, the limit is called an endpoint of Γ. Note that an arc Γ can have none, one or two endpoints and that an endpoint is not necessarily included in the arc. In the case that I is a closed interval, i.e. $I = [a, b]$, we denote by

$$|\Gamma| = \sup_{(t_0,\dots,t_n)\in\mathcal{P}} \sum_{i=1}^{n} |\phi(t_i) - \phi(t_{i-1})|,$$

the length of Γ. Here, $\mathcal{P} = \{(t_0, ..., t_n);\ n \in \mathbb{N},\ a = t_0 < t_1 < ... < t_n = b\}$ is the set of all partitions of the interval $[a, b]$. If I is not closed, the length of Γ is defined by

$$|\Gamma| = \sup_{[c,d]\subseteq I} |\phi([c, d])| \,.$$

If $|\Gamma| < \infty$, Γ is called a rectifiable arc. Note that for rectifiable arcs, the concept of non-tangential convergence almost everywhere is still meaningful. In fact, assume $\tilde{\phi}\colon \tilde{I} \to \Gamma$ is the arc length parametrization of Γ. Then $\tilde{\phi}$ is Lipschitz continuous and therefore differentiable almost everywhere on $\tilde{I}$ by Rademacher's Theorem. If Γ is equipped with the Lebesgue length measure (see chapter 1 in [**20**]), this means that we can assign the tangential direction $t(z) = \tilde{\phi}'(\tilde{\phi}^{-1}(z))/\|\tilde{\phi}'(\tilde{\phi}^{-1}(z))\| \in \mathbb{C}$ to almost every $z \in \Gamma$. For such z and $0 < \alpha < \pi/2$, the left and right α-cones at z may be defined by

$$\Gamma^+_\alpha(z) = z + t(z)\mathrm{i} \cdot \Gamma_\alpha \qquad \text{and} \qquad \Gamma^-_\alpha(z) = z + t(z)(-\mathrm{i}) \cdot \Gamma_\alpha.$$

Given complex-valued functions f and g defined on Γ and $\mathbb{C}\backslash\Gamma$, we then say that g converges non-tangentially from the left resp. right to f at z if

$$\text{(A.3)} \qquad \lim_{z'\to z,\ z'\in\Gamma^+_\alpha(z)} g(z') = f(z) \qquad \text{resp.} \qquad \lim_{z'\to z,\ z'\in\Gamma^-_\alpha(z)} g(z') = f(z)$$

for every $0 < \alpha < \pi/2$.
Next, we introduce contours that consist of more than one single arc. A subset $\Gamma \subseteq \mathbb{C}$ is a composed curve, if it is connected and can be written as a finite union $\Gamma = \bigcup_{i=1}^N \Gamma_i$, where each Γ_i is a arc and for $i \neq j$, Γ_i and Γ_j have at most endpoints in common. In addition, a composed curve is called an oriented composed curve, if the Γ_i's in the above representation are oriented.
Suppose $\Gamma \subseteq \mathbb{C}$ is a composed curve and $r > 0$. For any complex number $z \in \mathbb{C}$, we denote by $D(z, r) = \{y \in \mathbb{C};\ |y - z| < r\}$ the open disc with radius r centered at z. Then $\Gamma \cap D(0, r)$ can be written as an at most countable union of arcs. We say that $\Gamma \cap D(0, r)$ is rectifiable, if each of these arcs is rectifiable and the sum of the lengths is finite. In the case that $\Gamma \cap D(0, r)$ is rectifiable for every $r > 0$, we call Γ a locally rectifiable composed curve.

We equip any locally rectifiable composed curve $\Gamma \subseteq \mathbb{C}$ with the Lebesgue length measure and write $|A|$ for the measure of a measurable subset A. Note that, if Γ is oriented, the above implies that we can define a tangential direction $t(z) \in \mathbb{S}^1$ for almost every $z \in \Gamma$. This means the concept of non-tangential convergence almost everywhere can be formulated using (A.3) for oriented locally rectifiable composed curves as well.

In the following, we will define a class of contours, that may also contain the infinite point ∞ of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and introduce the so-called Carleson condition. This assumption turns out to be essential for the Cauchy operator to have the properties known in the classical framework. A subset $\Gamma \subseteq \hat{\mathbb{C}}$ is called a Carleson curve, if it is connected, $\Gamma \cap \mathbb{C}$ is a locally rectifiable composed curve and the Carleson condition

$$\sup_{z \in \mathbb{C}} \sup_{r>0} \frac{|\Gamma \cap D(z,r)|}{r} < \infty$$

is satisfied. Furthermore, we define the class $\mathcal{J}$ by

$$\mathcal{J} := \{\Gamma \subseteq \hat{\mathbb{C}};\ \Gamma \text{ is a Carleson curve and homeomorphic to the unit circle } \mathbb{S}^1\}.$$

Finally, we are ready to the introduce the Carleson jump contours, for which results similar to the ones for $\mathbb{R}$ can be obtained. A set $\Gamma \subseteq \hat{\mathbb{C}}$ is called a Carleson jump contour if it is connected, $\Gamma \cap \mathbb{C}$ is an oriented composed curve and $\hat{\mathbb{C}} \backslash \Gamma = D_+ \cup D_-$, where D_+ and D_- are disjoint, open subsets of $\hat{\mathbb{C}}$ with the following properties:

(i) $\Gamma = \partial D_+$ and $\Gamma = -\partial D_-$, meaning Γ is the positively oriented boundary of D_+ and the negatively oriented boundary of D_-.
(ii) D_+ and D_- each have finitely many, simply connected components in $\hat{\mathbb{C}}$.
(iii) If D is a component of D_+ or D_-, then $\partial D \in \mathcal{J}$.

If Γ is a Carleson jump contour, we can always write $\Gamma \cap \mathbb{C}$ as a finite union of oriented locally rectifiable arcs. But this already implies that a tangential direction $t(z) \in \mathbb{S}^1$ exists for almost every $z \in \Gamma \cap \mathbb{C}$, so the definition (A.3) of non-tangential convergence almost everywhere makes sense also for Carleson jump contours.

Theorem A.4. *Assume Γ is a Carleson jump contour and $\phi \in L^2(\Gamma)$. Then $\phi(s)/(s-z) \in L^1(\Gamma)$ for all $z \in \mathbb{C} \backslash \Gamma$ and the Cauchy integral*

$$\mathcal{C}\phi\,(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\phi(s)}{s-z}\, ds, \qquad z \in \mathbb{C}\backslash\Gamma,$$

defines a holomorphic function on $\mathbb{C}\backslash\Gamma$. Furthermore, there is a function $\mathcal{C}_+\phi$ (resp. $\mathcal{C}_-\phi$) in $L^2(\Gamma)$ such that $\mathcal{C}\phi$ converges non-tangentially from the left (resp. right) to $\mathcal{C}_+\phi$ (resp. $\mathcal{C}_-\phi$) at almost every $z \in \Gamma$. The corresponding mappings $\mathcal{C}_\pm \colon L^2(\Gamma) \to L^2(\Gamma)$, $\phi \mapsto \mathcal{C}_\pm\phi$ are bounded linear operators on $L^2(\Gamma)$ and the decomposition formula

$$\mathcal{C}_+\phi - \mathcal{C}_-\phi = \phi$$

holds true for every $\phi \in L^2(\Gamma)$.

Proof. This was shown in [**13**]. □

A.3. Representation of functions by Cauchy integrals

Another problem that arises naturally in the study of Riemann–Hilbert problems and Cauchy operators is to find necessary and sufficient conditions under which a function on $\mathbb{C}\backslash L$ can be represented by a Cauchy integral. In the framework of L^2 functions, the answer to these questions lies in the so-called Smirnoff spaces. The particular situation of Carleson jump contours has been treated in [**13**] and the below definitions and results have been collected from there.

A set $\Gamma \subseteq \mathbb{C}$ is called a Jordan curve, if there exists a homeomorphism $\phi\colon \mathbb{S}^1 \to \Gamma$ from the unit sphere $\mathbb{S}^1$ onto Γ. Note that any Jordan curve is a composed curve consisting of the two arcs $\phi(\{z \in \mathbb{S}^1;\ \mathrm{Im}(z) \geq 0\})$ and $\phi(\{z \in \mathbb{S}^1;\ \mathrm{Im}(z) \leq 0\})$. If both these arcs are rectifiable, we say that Γ is a rectifiable Jordan curve.

A set $D \subseteq \hat{\mathbb{C}}$ is a domain in $\hat{\mathbb{C}}$, if it is both open and connected with respect to the topology of $\hat{\mathbb{C}}$. Step by step, we now will define the Smirnoff classes $E^2(D)$ and $\dot{E}^2(D)$ for special types of domains in $\hat{\mathbb{C}}$. First, we consider domains D in $\hat{\mathbb{C}}$ that are bounded by a rectifiable Jordan curve. To this end, assume $\Gamma \subseteq \mathbb{C}$ is a rectifiable Jordan curve oriented counterclockwise. Then $\hat{\mathbb{C}}\backslash\Gamma$ is the disjoint union of two domains in $\hat{\mathbb{C}}$. We write D_- for the domain containing ∞ and refer to it as the exterior component. The other one will be denoted by D_+ and is called the interior component. A complex-valued function f on D_+ is said to be in the Smirnoff class $E^2(D_+)$, if it is analytic and there exists a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of rectifiable Jordan curves in D_+, such that every compact subset $K \subseteq D_+ \subseteq \hat{\mathbb{C}}$ is surrounded by Γ_n for $n \geq n_K$ and

$$\sup_{n\in\mathbb{N}} \int_{\Gamma_n} |f(z)|^2\, |dz| \,<\, \infty. \tag{A.4}$$

Similarly, a complex-valued function f on D_- belongs to the Smirnoff class $E^2(D_-)$, if it is analytic on D_- and there exists a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of rectifiable Jordan curves in D_-, such that every compact subset $K \subseteq D_- \subseteq \hat{\mathbb{C}}$ lies outside Γ_n for $n \geq n_K$ and (A.4) holds.
Now let $D \subseteq \hat{\mathbb{C}}$ be a domain in $\hat{\mathbb{C}}$ with $\partial D \in \mathcal{J}$. If $\infty \notin \partial D$, then ∂D is a Jordan curve and therefore a bounded subset of $\mathbb{C}$. By definition of $\mathcal{J}$, ∂D is a locally rectifiable, composed curve as well. But this implies ∂D is a rectifiable Jordan curve. Hence, D equals either the interior or exterior component of $\Gamma = \partial D$ and $E^2(D)$ is already defined by the above. If $\infty \in \partial D$, we fix $z_0 \in \mathbb{C}\backslash\partial D$ and define a Möbius transformation by

$$\phi(z) = \frac{1}{z - z_0}, \qquad z \in \hat{\mathbb{C}}.$$

Then ϕ is a homeomorphism of $\hat{\mathbb{C}}$ onto itself, which means $\phi(D)$ is a domain in $\hat{\mathbb{C}}$ with $\partial(\phi(D)) = \phi(\partial D))$. In addition, $\mathcal{J}$ is invariant under all linear fractional transformations, which means $\partial(\phi(D))$ is in $\mathcal{J}$ and $\infty \notin \partial(\phi(D))$. We can thus introduce the Smirnoff class $E^2(D)$ as the space of all complex-valued functions f on D, such that $f \circ \phi^{-1} \in E^2(\phi(D))$. $E^2(D)$ does not depend on the particular choice of z_0 and is therefore well-defined.
Let D be a domain in $\hat{\mathbb{C}}$ with $\partial D \in \mathcal{J}$. The subspace containing all functions $f(z)$ in $E^2(D)$ with $zf(z) \in E^2(D)$ is called the restricted Smirnoff class and will

be denoted by $\dot{E}^2(D)$. If $\infty \in D$, this definition implies that $\dot{E}^2(D)$ contains exactly those Smirnoff functions that vanish at infinity. Now suppose $D \subseteq \hat{\mathbb{C}}$ is a disjoint union of finitely many domains D_1,..., D_N in $\hat{\mathbb{C}}$ with $\partial D_i \in \mathcal{J}$ for every $i = 1, ..., N$. Then the restricted Smirnoff class $\dot{E}^2(D)$ is the space of all complex-valued functions f on D with $f_{|D_i} \in \dot{E}^2(D_i)$ for all $i = 1, ..., N$.

THEOREM A.5. *Suppose $\Gamma \subseteq \hat{\mathbb{C}}$ is a Carleson jump contour. Then the following statements hold true:*

(i) For every $f \in \dot{E}^2(\hat{\mathbb{C}}\backslash\Gamma)$, there exist functions f_+ and f_- in $L^2(\Gamma)$ such that f converges non-tangentially from the left to f_+ and non-tangentially from the right to f_- at almost every $z \in \Gamma$. Furthermore,

$$f = \mathcal{C}(f_+ - f_-), \qquad f \in \dot{E}^2(\hat{\mathbb{C}}\backslash\Gamma).$$

(ii) If $\phi \in L^2(\Gamma)$, then $\mathcal{C}\phi$ is contained in the restricted Smirnoff class $\dot{E}^2(\hat{\mathbb{C}}\backslash\Gamma)$.

PROOF. This is one of the main results proven in [**13**]. □

Finally, the next lemma provides a sufficient condition for functions to be in the restricted Smirnoff class.

LEMMA A.6. *Suppose $D \subseteq \hat{\mathbb{C}}$ is a domain in $\hat{\mathbb{C}}$ with $\partial D \in \mathcal{J}$ and f is a holomorphic function on D. If there exists a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of curves of class $\mathcal{J}$ in D, such that every compact subset $K \subseteq D \subseteq \hat{\mathbb{C}}$ is surrounded by Γ_n for $n \geq n_K$ and*

$$\sup_{n\in\mathbb{N}} \int_{\Gamma_n} |f(z)|^2 \, |dz| \; < \; \infty,$$

then f is in the restricted Smirnoff class $\dot{E}^2(D)$.

PROOF. See [**13**], Lemma 3.7. □

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