

Singularity Theorems Beyond C^2

A Status Report

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FWF Österreichischer
Wissenschaftsfonds

excellent = austria

Outline

- 1 Singularity theorems: A brief recap
- 2 Beyond C^2 : Why, what, and how
- 3 The analytic approach
- 4 Hawking's theorem for Lipschitz metrics
- 5 Hawking's theorem for Hölder metrics with L^p -curvature
- 6 Outlook

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The singularity theorems in General Relativity

About

- rigorous results in Lorentzian differential geometry
- physically reasonable ass. (collapse, expanding univ.) $\leadsto$ singularities
- singularity $\sim$ incomplete causal geodesic
- establish incompleteness as generic feature of GR

Why should you care?

- integral part of GR, especially of causality theory
- Roger Penrose's 2020 Nobel Prize in Physics

Here: Focus on *low regularity* extensions

- *causality theory* for metrics g (locally Lipschitz) continuous
- *geodesic focusing* for $g \in C^1$; loc. Lip.; $W^{1,p}$ with L^p -curvature

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The structure of the singularity theorems

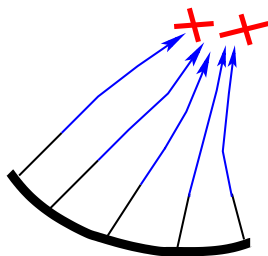
Pattern theorem (Senovilla, 1998)

A spacetime (M, g) is singular if it satisfies:

- (I) A suitable initial condition,
- (E) a curvature/energy condition,
- (C) a causality condition.

- (I) $\leadsto$ causal geodesics start focusing
- (E) $\leadsto$ focusing goes on (Raychaud./Riccati)
 - $\leadsto$ focal/conjugate point
 - $\leadsto$ geos. stop maximising
- (C) $\leadsto$ there are maximising causal geos

Resolution: some causal geodesics
stop existing before (first) conjugate point



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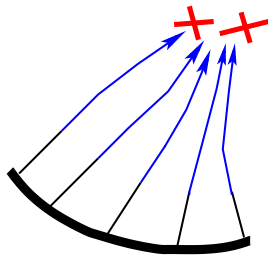
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geometric conditions to be fed
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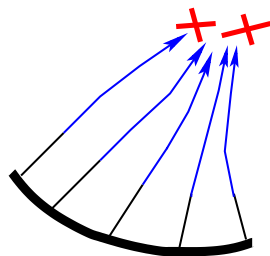
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The classical theorems

Thm. (Penrose, 65) Spacetime is future null incomplete if

- (E) $\text{Ric}(X, X) \geq 0$ for all X null (NEC)
- (C) there is a non-compact Cauchy hypersurface Σ
- (I) there is a future trapped surface S ($\theta^\pm < 0$)

Thm. (Hawking, 67) Future timelike incompleteness ($\tau_\Sigma \leq \frac{n-1}{|b|}$) if

- (E) $\text{Ric}(X, X) \geq 0$ for all X timelike (SEC),
- (C) there is a compact spacelike Cauchy surface Σ with
- (I) everywhere positive future convergence, resp. $\mathcal{H} \leq b < 0$

Thm. (Hawking-Penrose, 70) Improvements

- (E) (SEC) and genericity (C) chronological (*only!*)
- (I) cp. achronal set w/o. edge, trapped surface/point/submanifold

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One-sided versions (Totschnig, 93), (Andersson-Mars-Metzger-Simon, 09)

- (I) S separates Σ into $\Sigma \setminus S = \Sigma^- \cup \Sigma^+$ w. “outer” part Σ^+ non-compact
 - and only $\theta^+ < 0$, or even
 - S a stable MOTS ($\theta^+ = 0$) under genericity

Gannon-Lee Theorem (Gannon, 75), (Lee, 76)

- (I) Σ asymptotically regular and S non-simply connected

(Nontrivial topology $\Rightarrow$ null incompleteness.)

Higher-dimensional versions: (Costa e Silva, 10), (Costa e Silva-Minguzzi, 20)

Corrections in non-global hyperbolicity: (Galloway, 83), (Galloway-Ling, 22)

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Analytic core: curvature bounds give geodesic focusing

- $p \in M$, $v \in T_p M$ unique maximally extended geodesic γ on I_v
- γ *stops maximising* time separation after first *conjugate point*
($\sim$ almost meeting point of geodesics)
- conjugate point $\iff$ *Expansion* $\theta \rightarrow -\infty$
- θ obeys *Raychaudhuri equation*

$$\dot{\theta} = -\text{Ric}(\dot{\gamma}, \dot{\gamma}) - \text{tr}(\sigma^2) - \frac{\theta^2}{3}$$

- second/shear term always non-positive
- Assume energy condition: $\text{Ric}(\dot{\gamma}, \dot{\gamma}) \geq 0$ (SEC) or (NEC)
- Assume initial condition: $\theta(0) < 0$

$\implies \theta \rightarrow -\infty$ for some t in $[0, -C(n)/\theta(0)) \implies$ conjugate point

Message: Bounds on Ricci curvature bound length of maximisers.

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Why low regularity matters: Nature of singularities in GR

[Hawking-Ellis,73], [Senovilla,98]

- by definition: singularity $\sim$ causal geodesic incompleteness
 - + allows to prove geometric results, even more objectionable
 - +/- singularities *not in* spacetime, rather boundary point
 - hard to relate to curvature blow up; trivially cutting out pts.

$\leadsto$ question of extensions

- ▶ causal geodesic completeness $\implies$ inextendability

$\leadsto$ extendability as a source of incompleteness has to be avoided

- ▶ apply thms. to maximally extended spacetimes

!!! Here the **regularity of extensions** becomes essential

- ▶ classical thms. only work for C^2 -extensions
- ▶ but extensions are ok as long as field equations make sense
- ▶ harmless drop of regularity vs. real singularity

Message: Lack of low-regularity thms. undermines their significance: singularity/incompleteness could be avoided by a drop in regularity.

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- classical: $g \in C^2 \leadsto$ Ricci curvature $\boxed{\text{Ric} \sim \partial^2 g (\partial g)^2}$ continuous
- $g \in C^{1,1}$: finite jump of matter variables, curvature still L^∞
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Extensions of causality theory in a nutshell

- systematic studies since [Chrusciel-Grant,12], [Fathi-Siconolfi,12]

✓ $C^{1,1}$: bulk of causality works [Minguzzi,15], [KSSV,14]

✓ $C^{0,1}$: most things work [Minguzzi,19] although **convexity fails !X!**

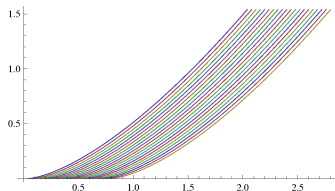
!X! below Lipschitz: some *fundamentals break down* [CG,12]

(1) push up principle fails

$$(J^+ \circ I^+) \not\subseteq I^+$$



(2) light cones bubble up



(3) The future I^+ needs not to be open [GKS,20]

- ✓ some topological features are more robust: Avez-Seifert [Sämann,16]
cone structures [Bernard-Suhr,18], [Minguzzi,19]

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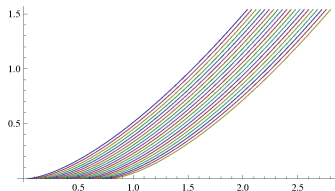
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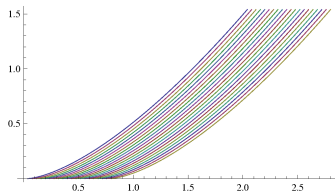
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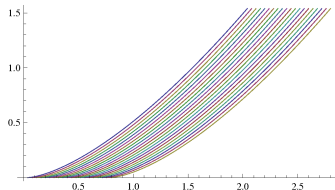
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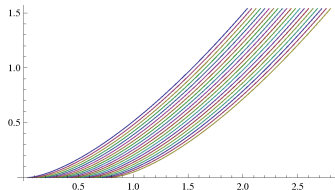
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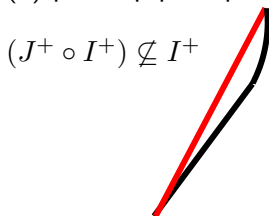
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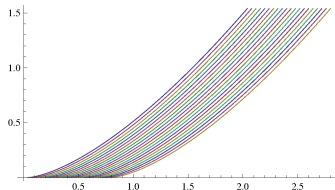
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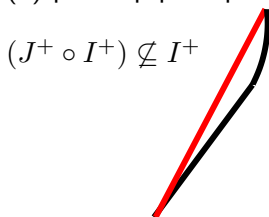
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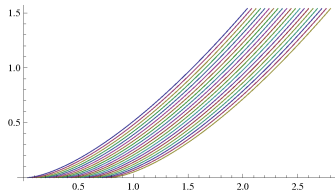
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Alternative settings: Beyond the analytic route

- **Causality-only versions** [Minguzzi,19]
 - ▶ closed cone structures; the *causal core* of the theorems
 - ▶ analytic input only to produce sets of prescribed causality
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Causal Penrose Theorem

[Minguzzi,19]

Let (M, C) be a globally hyperbolic closed cone structure admitting a non-compact stable Cauchy hypersurface.

Then there are no compact future trapped sets S .

$(S \neq \emptyset \text{ with } E^+(S) \text{ compact})$

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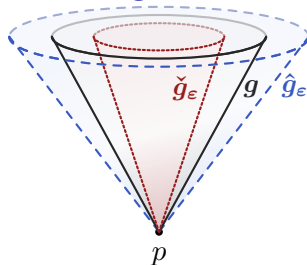
The foundation: causality adapted smoothing

Chartwise convolution of the metric

$$\begin{aligned} g_\epsilon(x) &= g \star_M \rho_\epsilon(x) \\ &= \sum \chi_i(x) \psi_i^* \left((\psi_i^*(\zeta_i \cdot g)) * \rho_\epsilon \right)(x) \end{aligned}$$

tweak to sandwich light cones [CG,12]

$$\check{g}_\epsilon \prec g \prec \hat{g}_\epsilon$$



Lemma (Convergence for continuous g)

[Chruciel-Grant++]

- basic convergence: $\check{g}_\epsilon, \hat{g}_\epsilon \rightarrow g$ locally uniformly; also for inverses
- enhanced convergence: $g \in W^{s,p}$ ($1 \leq p < \infty$), convergence in $W^{s,p}$
- strong compatibility of distinct regularisations:
 $\check{g}_\epsilon - g_\epsilon \rightarrow 0$ in C^∞ ; plus inverses $\leadsto \boxed{\text{Ric}[\check{g}_\epsilon] - \text{Ric}[g_\epsilon] \rightarrow 0}$ in C^∞ ;
 same for $\hat{g}_\epsilon$
- quantified convergence: $\|\nabla_h^k(\check{g}_\epsilon - g_\epsilon)\|_{\infty,K} \leq C\epsilon$ ($k = 0, 1$);
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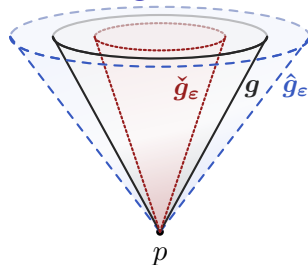
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Non-smooth focusing: The rough guide

- ① Formulate suitable
low regularity/distributional energy/curvature conditions
for $\text{Ric}[g]$, $H[g]$ and rough g
- ② Derive (typically weaker than classical) surrogate conditions
for $\text{Ric}[\check{g}_\epsilon]$, $H[\check{g}_\epsilon]$ (on compact sets)
...analysis of convolution
- ③ still show smooth focusing for $\check{g}_\epsilon$ -geodesics
...extend focusing analysis
- ④ show that (a suitable notion of) geos of the rough g stop maximising
e.g. via approximating g -geodesics by $\check{g}_\epsilon$ -geodesics in C^1
...use ODE theory
- ⑤ contradict global causality properties of g
...extend causality

Non-smooth focusing: Following the rough guide for Ric

- Start with distributional condition $\text{Ric}[g] \geq 0$
- **Problem:** $\text{Ric}[\check{g}_\epsilon] \rightarrow \text{Ric}[g]$ only distributionally
 $\leadsto$ cannot carry $\text{Ric}[g] \geq 0$ through construction
- **Solution:**
 - ▶ convolution preserves sign: $\text{Ric}[g] \star_M \rho_\epsilon \geq 0$
 - ▶ Friedrichs-Lemma (commutators are better than you think!)

$$\|\text{Ric}[g \star_M \rho_\epsilon] - \text{Ric}[g] \star_M \rho_\epsilon\|_{\infty, K} \rightarrow 0$$

- ▶ compatibility of distinct regularisations: $\|\text{Ric}[\check{g}_\epsilon] - \text{Ric}[g_\epsilon]\|_{\infty, K} \rightarrow 0$

Lemma (Almost preservation of energy condition) [Graf,20]

The distributional strong energy condition for $g \in C^1$

$$\text{Ric}[g](\mathcal{X}, \mathcal{X}) \geq 0 \text{ in } \mathcal{D}' \text{ for all } g\text{-timelike vector fields } \mathcal{X} \quad (\text{DSEC})$$

gives: for all $\delta > 0$, all $X \in K \subseteq TM$ with $\check{g}_\epsilon(X, X) = -1$ and all small ϵ

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Enhanced ODE-Theory and the C^1 -theorems

Proposition (Approximating maximising geodesics) [Graf,20]

Let (M, g) be globally hyperbolic with $g \in C^1$. Then for any $p \leq q$ there exists at least one maximizing causal geodesic from p to q that is obtained as a C^1 -limit of a sequence of $\check{g}_\epsilon$ -maximizing $\check{g}_\epsilon$ -causal $\check{g}_\epsilon$ -geodesics.

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Then $\sup_M \tau_\Sigma \leq \frac{n-1}{|b|}$.

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Lipschitz regularity: obstructions & strategy

jointly w. M. Calisti, M. Graf, E. Hafemann, M. Kunzinger

- **Problem:** below C^1 ODE-theory breaks down
 - ▶ geodesic equation has L^∞ -coefficients
 - ▶ no suitable ODE-theory to give continuous dependence on r.h.s. $\Gamma[\check{g}_\epsilon]$
 - ▶ no approximation of g -geodesics/maximisers by $\check{g}_\epsilon$ -geodesics $\leadsto$ classical *pointwise* focusing is unavailable
- **Solution:** two new ingredients
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Friedrichs-type arguments provide necessary L^1 -estimates

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Feeding the analytic result into the focusing machinery

Lemma (Curvature estimates) [CGHKS,25]

- $\text{Ric}[\check{g}_\varepsilon] - (\dot{\gamma}, \dot{\gamma}) \rightarrow 0$ in $L^1(K)$
- $\text{Ric}[\check{g}_\varepsilon](\dot{\gamma}, \dot{\gamma}) \geq n\kappa$ for some $\kappa < 0$

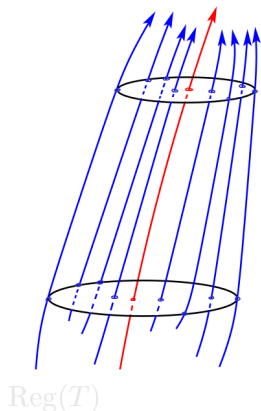
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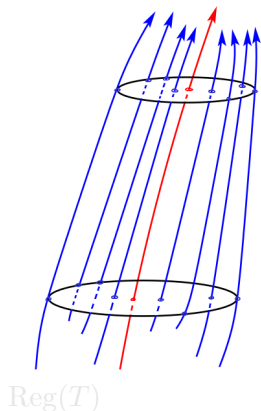
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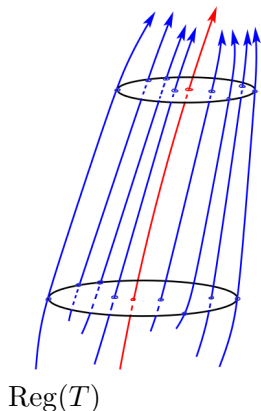
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$$\begin{aligned} \text{area}_g(\text{Reg}(T)) &\leq \limsup \text{area}_{\check{g}_\varepsilon}(\check{\text{e}}\text{Reg}(T)) \\ &\leq C \int_{\Omega} |\text{Ric}[\check{g}_\varepsilon] - (U, U)| \rightarrow 0 \end{aligned}$$

So $\text{area}_g \text{Reg}(T) = 0$; as good as a conjugate point



Feeding the analytic result into the focusing machinery

Lemma (Curvature estimates) [CGHKS,25]

- $\text{Ric}[\check{g}_\varepsilon]_-(\dot{\gamma}, \dot{\gamma}) \rightarrow 0$ in $L^1(K)$
- $\text{Ric}[\check{g}_\varepsilon](\dot{\gamma}, \dot{\gamma}) \geq n\kappa$ for some $\kappa < 0$

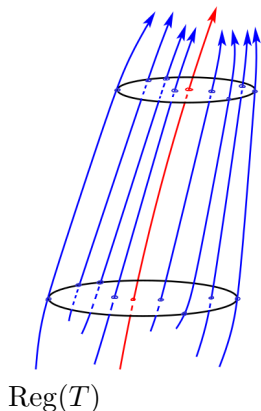
Look at set of good points in Cauchy surface
 $\text{Reg}(T) \ni x$ if geo. starting at x max. up to T

Volume estimate [Treude-Grant,13]

& segment inequality: line- $\int$ from volume- $\int$
[Graf-Kontou-Ohanyan-Schinnerl,25]
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relevant terms in $\text{Ric}[\check{g}_\epsilon] - \text{Ric}[g] \star_M \rho_\epsilon$

$$\underbrace{[(\psi_\beta)_* g_\epsilon]^{ij}}_{=: a_\epsilon} \underbrace{([\xi \partial_k ((\psi_\beta)_* g)_{lm}])^{*} \rho_\epsilon}_{=: f} - \underbrace{[(\psi_\beta)_* g]^{ij}}_{=: a} \underbrace{[\xi \partial_k ((\psi_\beta)_* g)_{lm}]}_{=: f} * \rho_\epsilon$$

Prove $a_\epsilon f_\epsilon - (af)_\epsilon \rightarrow 0$ in $W^{1,1}$ & bounded in $W^{1,\infty}$ on $K \Subset M$ for
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Write relevant term as integral op. [Braverman-Milatovic-Shubin,02]

$$K_\epsilon f(x) = \int k_\epsilon(x, y) f(y) dy = \int \partial_x \left((a(x) - a(y)) \rho_\epsilon(x - y) \right) f(y) dy$$

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Hawking Theorem for locally Lipschitz metrics, 1

Thm. (Lipschitz Hawking 1: time-separation bound) [CGHKS,24]

Let (M, g) be a globally hyperbolic spacetime with loc. Lipschitz g with:

- (i) $\text{Ric}_g(X, X) \geq 0$ distributionally for all timelike $X \in \mathfrak{X}(M)$;
- (ii) a smooth spacelike Cauchy hypersurface Σ with $\mathcal{H}[g] < b < 0$.

Then $\sup_M \tau_\Sigma \leq \frac{n-1}{|b|}$.

Here we use the *slab-mean curvature*:

- $g \in \text{Lip} \rightsquigarrow \nabla_{X_i} X_j$ only $L^\infty \rightsquigarrow$ no mean curvature H of null set Σ
- thicken Σ by flow out by timelike vector field X to neighbourhood A_Σ
- define $\mathcal{H}_{A_\Sigma}^X[g] = \sum_{i,j=1}^{n-1} G^{ij} g(\nabla_{X_i} X_j, N)$, $(G^{ij}) = (g(X_i, X_j))^{-1}$
- define bounds: $\mathcal{H}[g] < b$ if $\text{ess sup } \mathcal{H}_{A_\Sigma}^X[g] < b$
- independent of X , A_Σ and compatible with smooth case

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Hawking Theorem for locally Lipschitz metrics, 2

Thm. (Lipschitz Hawking 2: incompleteness)

[CGHKS,24]

Let (M, g) be a spacetime with locally Lipschitz g with

- (i) $\text{Ric}_g(X, X) \geq 0$ distributionally for all timelike smooth X
- (ii) a smooth *compact* spacelike hypersurface Σ with $\mathcal{H}[g] < b < 0$

Then there is an incomplete, future-directed timelike Filippov geodesic γ emanating orthogonally from Σ , with length $L(\gamma) \leq \frac{n-1}{|b|}$.

Here we use *Filippov's solution concept for ODEs*:

- replace ODE $\ddot{\gamma}^i = -\Gamma_{jk}^i[g]\dot{\gamma}^j\dot{\gamma}^k$ by a differential inclusion $\ddot{\gamma}^i \in \mathcal{F}(-\Gamma_{jk}^i[g]\dot{\gamma}^j\dot{\gamma}^k)$, a convex closed set constructed from image of the L^∞ -r.h.s. (e.g. by smoothing)
- Every $g \in C^{0,1}$ possesses (non-unique) F. geos which are $C^{1,1}$ [S,14]
- Every causal maximiser for g is a F. geo [Lange-Lytchak-Sämann,21]

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- 6 Outlook

Going below Lipschitz & a new key new tool

jointly w. M. Kunzinger, M. Reintjes, I. Vega-Gonzalez

- We now consider $g \in W^{1,p} \subseteq C^{0,\alpha}$ with $\alpha = 1 - n/p$ for $p > n$.
- Sounds dangerous due to causality break down below Lipschitz

Thm. (RT-regularisation)

[Reintjes-Temple,20–24]

On manifold $(\mathcal{M}, \mathcal{A})$ w. C^∞ -atlas $\mathcal{A}$, $\boxed{g \in W_{\mathcal{A}}^{1,p}}$ ($p > \max\{4, n\}$, $p < \infty$) there exists a C^∞ -atlas $\mathcal{A}'$, which is $W^{2,p}$ -compatible with $\mathcal{A}$, such that

$$g \in W_{\mathcal{A}'}^{2,p} \quad \text{if (and only if)} \quad \text{Riem}[g] \in L_{\mathcal{A}}^p.$$

- *elliptic regularity* in coordinates raises g 's regularity by one derivative
- L^p -curvature bound $\leadsto$ optimal regularity in $W^{2,p}$ -compatible atlas
- Semi-Riemannian analogue of classical result by [Kazdan-De Turk,81]

L^p -curvature boosts regularity of the metric

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The use of RT-regularisation: The basic idea

- start on **bad side**: $g \in W^{1,p}$, $\text{Riem} \in L^p$ ($p > n \geq 4$):
pose reasonable conditions here
 - change to **good side**: $g \in W^{2,p} \subseteq C^{1,\alpha} \subseteq C^1$
use all benefits here and change back to bad side
 - ▶ causality is ok on the good side,
so it was already ok on bad side since $I_{\mathcal{A}}^{\pm}(p) = I_{\mathcal{A}'}^{\pm}(p)$
 - ▶ notion of *RT-geodesics*: solving classical equation on good side,
gives C^1 -curves on the bad side (independent of choice of $\mathcal{A}'$)
 - ▶ use C^1 -theorems and easily boost regularity to $g \in W^{1,p}$, $\text{Riem} \in L^p$
 - **But there is no free lunch!**
 - ▶ C^∞ -Cauchy/trapped surface on bad side is only $C^{1,\alpha}$ on good side!
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Another mean curvature & a two step regularisation

- mean curvature of Σ on **bad side**: $\Sigma \in C^\infty$, $g \in W^{1,p}$:
 - ▶ Write $\Sigma = \mathcal{T}^{-1}(0)$ for a C^∞ -Cauchy temporal function by [Minguzzi,20]/[Bernard-Suhr,20]'s C^0 -version of [Bernal-Sánchez,06]
 - ▶ *smeared-out mean curvature*

$$\mathcal{H}_{g,\mathcal{T},U} := - \sum_{i,j=1}^{n-1} G^{ij} g(\nabla_{X_i} X_j, N) \in L^p(U)$$

$N \in W^{1,p}$ unit normal to slices $\mathcal{T}^{-1}(t)$, local frame $X_i \in W^{1,p}$

- ▶ *mean curvature bound* $H_\Sigma[g] < b$ if $\exists \mathcal{T}, U$: $\text{ess sup } \mathcal{H}_{g,\mathcal{T},U} < b$
- RT-regularisation to **good side**: $g \in W^{2,p}$ but $\Sigma \in W^{2,p}$, $\mathcal{T} \in W^{2,p}$
 - ▶ mean curvature bound simply translates
- **manifold convolution** to enter into [Graf,20]'s C^1 -proofs and contradict classical Hawking for $\check{g}_\epsilon$
 - ▶ need *refined analysis of manifold convolution* to show

$$\mathcal{H}_{\check{g}_\epsilon, \mathcal{T}_\epsilon, U} - \mathcal{H}_{g, \mathcal{T}, U} \star_{\mathcal{M}} \rho_\epsilon \rightarrow 0 \quad \text{in } L^\infty(U).$$

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Another mean curvature & a two step regularisation

- mean curvature of Σ on **bad side**: $\Sigma \in C^\infty$, $g \in W^{1,p}$:
 - ▶ Write $\Sigma = \mathcal{T}^{-1}(0)$ for a C^∞ -Cauchy temporal function by [Minguzzi,20]/[Bernard-Suhr,20]'s C^0 -version of [Bernal-Sánchez,06]
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$$\mathcal{H}_{g,\mathcal{T},U} := - \sum_{i,j=1}^{n-1} G^{ij} g(\nabla_{X_i} X_j, N) \in L^p(U)$$

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The analytic heart: another Friedrichs-type estimate

$$\mathcal{H}_{\check{g}_\epsilon, \mathcal{T}_\epsilon, U} - \mathcal{H}_{g, \mathcal{T}, U} \star_M \rho_\epsilon \rightarrow 0 \quad \text{in } L^\infty(U) \text{ for } p > 2n$$

Write $\boxed{\mathcal{H}_{g, \mathcal{T}, U} = \Phi(\mathbf{U}) \cdot w}$ with $\Phi \in C^\infty$, $\mathbf{U} = (g, \nabla^h g, d\mathcal{T}, (X_i))$

and the **bad term** $w = \text{Hess}(\mathcal{T})$: $\|w_\epsilon\|_\infty = \|w \star_M \rho_\epsilon\|_\infty = O(\epsilon^{-n/p}) \rightarrow \infty$

$$\begin{aligned} & \|\Phi(\check{\mathbf{U}}_\epsilon) \cdot w_\epsilon - (\Phi(\mathbf{U}) \cdot w) \star_M \rho_\epsilon\|_{L^\infty(K)} \\ & \leq \underbrace{\|\Phi(\check{\mathbf{U}}_\epsilon) - \Phi(\mathbf{U}_\epsilon)\|_{L^\infty(K)}}_{O(\epsilon) \text{ quantified conv.}} \underbrace{\|w_\epsilon\|_{L^\infty(K)}}_{O(\epsilon^{-n/p})} \dots O(\epsilon^{1-n/p}) \\ & + \underbrace{\|\Phi(\mathbf{U}_\epsilon) - \Phi(\mathbf{U}) \star_M \rho_\epsilon\|_{L^\infty(K)}}_{O(\epsilon^{2\alpha}) \text{ nonlin. commutator est.}} \underbrace{\|w_\epsilon\|_{L^\infty(K)}}_{O(\epsilon^{-n/p})} \dots O(\epsilon^{2-3n/p}) \\ & + \underbrace{\|\Phi(\mathbf{U}) \star_M \rho_\epsilon \cdot w \star_M \rho_\epsilon - (\Phi(\mathbf{U}) \cdot w) \star_M \rho_\epsilon\|_{L^\infty(K)}}_{O(\epsilon^{1-2n/p}) \text{ Friedrichs-type estimate}} \\ & \rightarrow 0 \quad \text{if } p > 2n \end{aligned}$$

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Hawking Theorem for Hölder metrics with L^p -curvature

Thm. ($W^{1,p}$ -Hawking thm. with L^p -bounded curvature, 1) [KRSV,26]

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime with metric $g \in W^{1,p}$ and curvature $\text{Riem}[g] \in L^p$ for some $p > 2n$ with:

- (i) $\text{Ric}(X, X) \geq 0$ a.e. for all smooth timelike vector fields X ;
- (ii) a smooth spacelike Cauchy hypersurface Σ with $\mathcal{H}[g] < b < 0$.

Then $\sup_M \tau_\Sigma \leq \frac{n-1}{|b|}$.

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Outlook

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Some Literature



M. Calisti, M. Graf, E. Hafemann, M. Kunzinger, R. Steinbauer. *Hawking's singularity theorem for Lipschitz Lorentzian metrics*. CMP **406**(9) (2025)



M. Graf. *Singularity theorems for C^1 -Lorentzian metrics*. CMP **378**(2) (2020)



M. Graf, E.-A. Kontou, A. Ohanyan, B. Schinnerl. *Hawking-type singularity theorems for worldvolume energy inequalities*. AHP **26** (2025).



M. Kunzinger, M. Reintjes, R. Steinbauer, I. Vega-González. *The Hawking singularity theorem for Hölder continuous metrics with L^p -bounded curvature*. Preprint, arXiv:2604.27023 (2026)



E. Minguzzi. *Causality theory for closed cone structures with applications*. Rev. Math. Phys. **31**(5) (2019)



M. Reintjes, B. Temple. *The essential regularity of singular connections in geometry*. Preprint, arXiv:2412.08928 (2024)



R. Steinbauer. *The singularity theorems of general relativity and their low regularity extensions*. Jahresber. Dtsch. Math.-Ver. **125**(2) (2023)