A new Lorentzian Geometry

Roland Steinbauer

Fakultät für Mathematik

universität wien

Mathematisches Kolloquium

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excellent = austria

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Regularity determines basic geometric features

Example 1: Walking on a sphere vs. walking on a cube.

Sphere: Walking along a geodesic/meridian is locally always the shortest path.

Cube: It is always shorter to deviate to the right face than to go along the edge.

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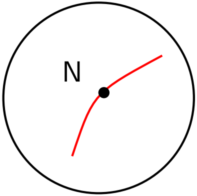

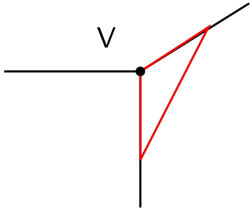

Sphere: Walking along a geodesic/meridian is locally always the shortest path.

Cube: It is always shorter to deviate to the right face than to go along the edge.

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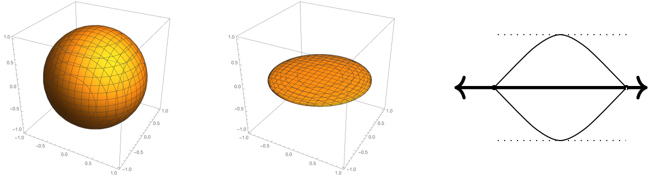
Example 2: Squeezing the sphere. [Hartman-Wintner, 52]



Convexity fails for metrics of Hölder regularity $g \in C^{1,\alpha}$ $(\alpha < 1)$:

Equator still geodesic but it is always shorter to deviate into hemispheres

Realised by $M := (-1,1) \times \mathbb{R}$ with

$$g_{(x,y)} = (1 - |y|^\lambda)dx^2 + dy^2 \quad (1 < \lambda < 2)$$

Lorentzian version: [Sämann-S, 19]

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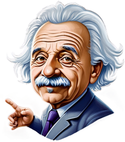
Lorentzian Geometry is the Language of General Relativity

GR, Einstein's theory of space, time & gravity:
Gravity is universal, hence prop. of surrounding spacetime
Curvature proportional to mass/energy content



$$Ric - \frac{1}{2} R\, g \;=\; \frac{8\pi G}{c^4}\, T \qquad \text{(E)}$$

Geometric description via a spacetime manifold (M, g)

- M ... $(4-$dimensional$)$ smooth manifold
- g ... *smooth Lorentzian metric* on M:
  non-degenerate scalar product g_p in each tangent space T_pM
  with signature $(-,+,+,+)$ not positive definite!

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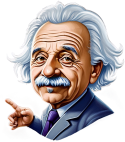

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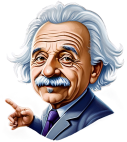

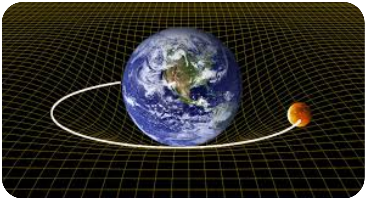

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Matter tells spacetime how to curve,
spacetime tells matter how to move.

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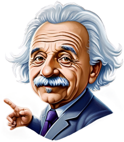

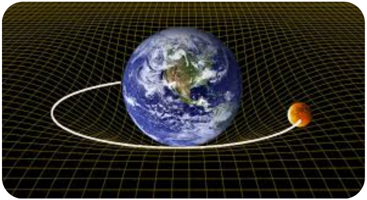

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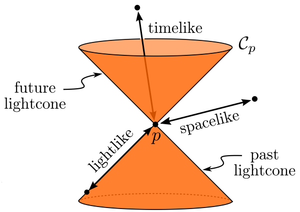

$$v \in T_pM, \quad g_p(v,v) \begin{cases} < 0 & \text{timelike} \\ = 0 & \text{lightlike/null} \\ \leq 0 & \text{causal} \\ > 0 & \text{spacelike} \end{cases}$$

extends naturally to
(loc. Lipschitz) curves

Causal relations between points $p, q \in M$

$p \ll q$:⇔ connected by future timelike curve
$p \leq q$:⇔ connected by future causal curve

Chronological/causal future/past

$I^+(p) := \{q \in M : \ p \ll q\}$
$J^+(p) := \{q \in M : \ p \leq q\}$
past versions: $I^-(p)$, $J^-(p)$
for $A \subseteq M$: $I^\pm(A)$, $J^\pm(A)$

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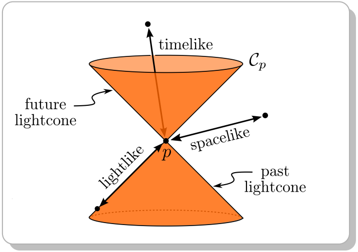

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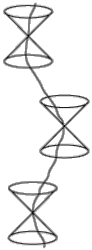

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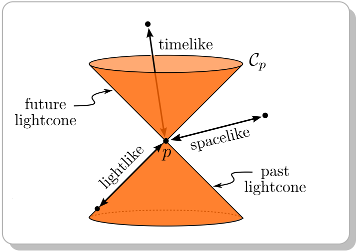

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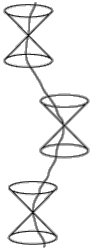

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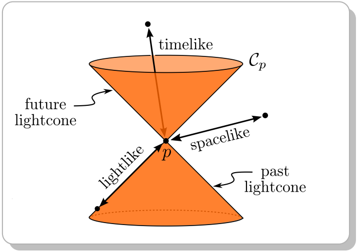

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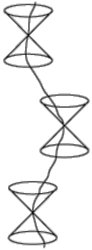

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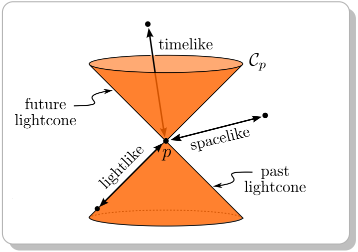

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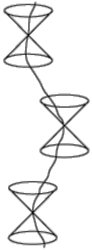

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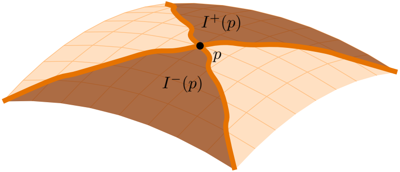

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Smooth Lorentzian Geometry is not enough

Generally from (E): $Ric - \frac{1}{2}Rg = \frac{8\pi G}{c^4}\,T$; $\quad Ric \propto \partial^2 g + (\partial g)^2$;
$T \notin C^0 \rightsquigarrow g \propto C^{1,1}$; below C^2 (smooth f.a.p.p.) non-smooth:=below C^2

Motivations from physics & mathematics

- *PDE/initial value* point-of-view (regularity in hyperbolic PDE!)
- physically relevant *models*: matched spacetimes

  [Senovilla, Mars, Sánchez-Pérez, Manzano, Ohanyan, S.,. . .]

  impulsive gravitational waves

  [Barrabés, Griffiths, Hogan, Podolský, Sämann, Švarc, S.. . .]
- approaches to *Quantum Gravity* (causal set theory, CDT,. . .)

"The discontinuities of the matter [...] which must be allowed [...] pose enormous problems [...]. To avoid this annoying problem though—despite it being completely fundamental!—we will implicitly assume that g is at least C^2." *[Garcia Parrado-Senovilla, CQG, 2005]*

But much has happened since!

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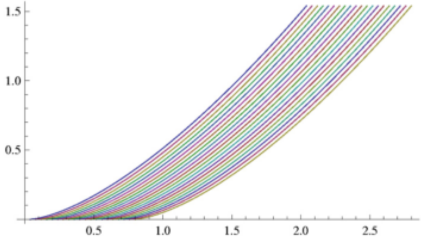
Regularity determines basic geometric features

Example 3: Lightcones bubble up. [Chruściel-Grant, 12]



Some null geodescis for $\alpha = 1/3$

$M = \mathbb{R}^2$, $g \in C^{0,\alpha}$ $(\alpha < 1)$

$$g = -(dt + (1 - |t|^\alpha)dx)^2 + dx^2$$

Non-unique null geodesics
$\rightsquigarrow$ null cone has full measure
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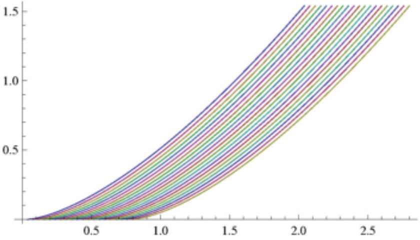

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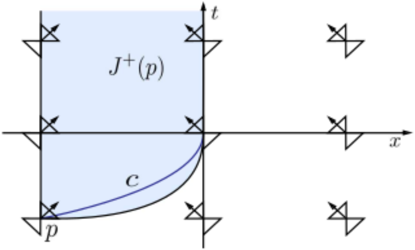
Ex. 4: The future is not always open. [Grant-Kunzinger-Sämann-S, 20]



$M = \mathbb{R}^2$, $g \in C^{0,\alpha}$ $(\alpha < 1)$

The blue curve c is timelike but reaches $\partial I^+(p)$
$\rightsquigarrow$ $I^+(p)$ *is not open*

Lightcones turn in non-Lipschitz way

What can be done: Approaches to non-smooth LG

Spacetime setting: smooth manifold but metric g below C^2 *(analytic)*

- distributional curvature: $g \in H^1 \cap L^\infty$ maximal w stable $\mathcal{D}'$-curvature
  [Geroch-Traschen, 87], [LeFloch-Mardare, 07], [Vickers-S, 09]
- non-linear distributional geometry (algebras of generalised functions)
  [Farkas, Grosser, Hanel, Hörmann, Kunzinger, Mayerhofer, Nigsch, Oberguggenberger, Spreitzer, Vickers, S. . . 01–]
- advanced regularisation techniques $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ [Chruściel-Grant, 12]
- causality for $g \in C^0$ [Sämann, 16], closed cone structures [Minguzzi, 19]

$\rightsquigarrow$ Singularity theorems in low regularity: $g \in C^{1,1}$, $g \in C^1$, $g \in C^{0,1}$
[Graf, Grant, Kunzinger, Ohayan, Schinnerl, Stojkovic, Vickers, S, 15–]
currently: $g \in W^{1,p}$, $R \in L^p$ [Kunzinger-Reintjes-Vega-S, 25–]

- C^0-inextendibility [Sbierski, 15-, Ling, 24, Monsani-Solanki-Prados-Sämann, ong.]

A new geometry: beyond the manifold setting *(synthetic)*

This will be the focus for the rest of the talk.

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How to detect curvature: A glimpse of Riemannian mflds

Riemannian manifold (M, h); curvature $R_{XY}Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$

Sectional curvature $\mathrm{Sec}(X, Y) = \dfrac{\langle R_{XY}Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}$

Theorem (Toponogov $\triangle$-comparison) $\quad \mathrm{Sec} \Longleftrightarrow$

For all (small) geodesic triangles $\triangle abc$ in (M, h) consider a comparison triangle $\triangle\bar{a}\bar{b}\bar{c}$ in the .
Then for all p, q on its sides and corresponding comparison points $\bar{p}$, $\bar{q}$

$$d_h(p, q)(\bar{p}, \bar{q}).$$

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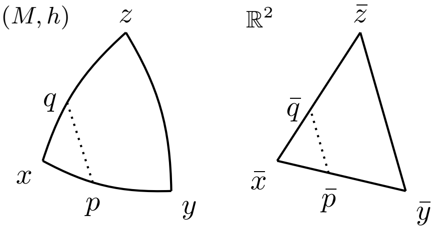
Theorem (Toponogov △-comparison) $\mathrm{Sec} \geq 0 \iff$

For all (small) geodesic triangles $\triangle abc$ in (M,h) consider a comparison triangle $\triangle \bar{a}\bar{b}\bar{c}$ in the Euclidean Plane $\mathbb{R}^2$.
Then for all p,q on its sides and corresponding comparison points $\bar{p}$, $\bar{q}$

$$d_h(p,q) \geq d_{\mathbb{R}^2}(\bar{p},\bar{q}).$$



Riemannian distance d_h

$$d_h(p,q) := \inf\{L_h(\gamma) : \gamma \text{ from } p \text{ to } q\}$$

w length of locally Lipschitz curves γ

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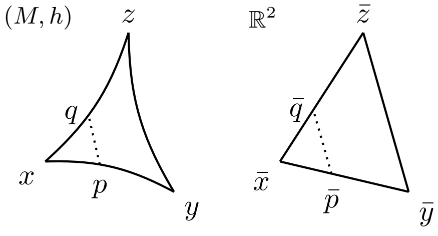

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Riemannian manifold (M,h); curvature $R_{XY}Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$

Sectional curvature $\mathrm{Sec}(X,Y) = \dfrac{\langle R_{XY}Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$

Theorem (Toponogov $\triangle$-comparison) $\quad \mathrm{Sec} \geq K \iff$

For all (small) geodesic triangles $\triangle abc$ in (M,h) consider a comparison triangle $\triangle\bar{a}\bar{b}\bar{c}$ in the 2D *model space of constant curvature K*. Then for all p,q on its sides and corresponding comparison points $\bar{p}$, $\bar{q}$

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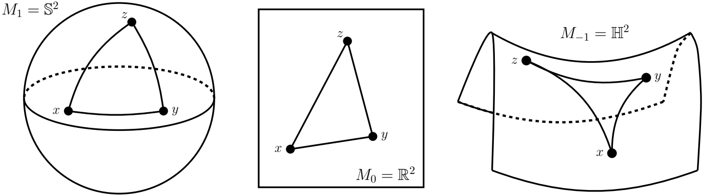
model spaces M_K of constant curvature K



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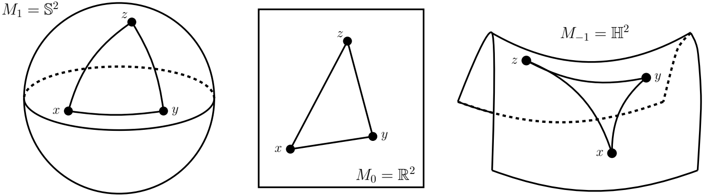

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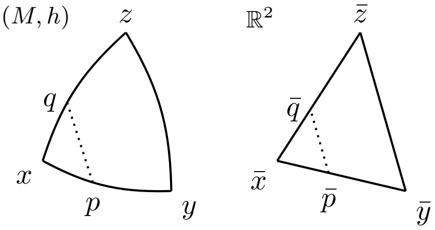

Triangle condition

- needs no manifold structure, only distances between pts.
- works on metric spaces, can be turned into definition

Sectional curvature bounds for metric spaces

Definition (Length space) (X,d) metric space w d intrinsic, i.e.

$$d(x,y) = \inf\{L_d(\gamma) : \gamma \text{ from } x \text{ to } y \text{ continuous}\}$$

Geodesics $\gamma : [0,1] \to X$ with $L_d(\gamma) = d(\gamma(0), \gamma(1))$

Definition (Synthetic curvature bounds)

A length space has curvature bounded by K if for all (small) *geodesic triangles* $\triangle abc$ and their *comparison triangles* $\triangle \bar{a}\bar{b}\bar{c}$ in the model space of curvature K and all points p, q on its sides and corresponding $\bar{p}$, $\bar{q}$

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Curvature bounded below / above: *Alexandrov spaces* / *CAT(K)-spaces*
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Definition ("Correct" curvature bounds) [Andersson-Howard, 98]

A smooth Lorentzian manifold has $\mathrm{Sec} \geq K$ if
spacelike sectional curvatures $\geq K$ and *timelike* sectional curvatures $\leq K$.

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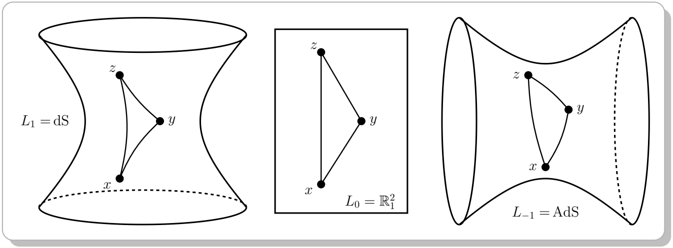

Lorentzian model spaces L_K

- $K > 0$: de Sitter
- $K = k$: Minkowski
- $K < 0$: anti-de S.

Go synthetic!

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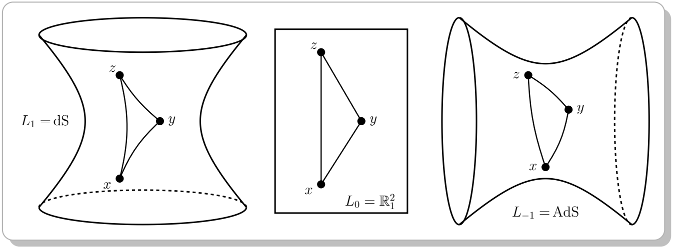

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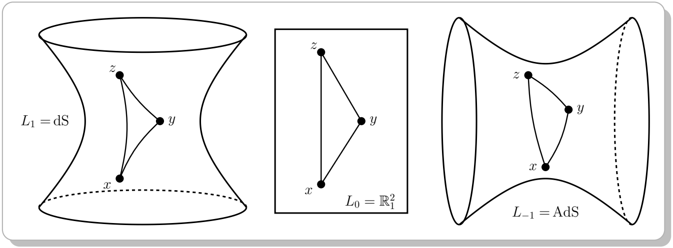

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How to go beyond Lorentzian manifolds?

	smooth	**synthetic**
pos. def.	Riemannian mflds	metric length spaces
Lorentzian	Lorentzian manifolds	???

What is the analogue of metric (length) spaces in the *Lorentzian setting*?

Serious issue

No natural distance function $\rightsquigarrow$ no direct entry into world of metric spaces

Lorentzian (pre-)length spaces [Kunzinger-Sämann, 18] based on causal spaces [Kronheimer-Penrose, 67] & the use of the time separation

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Lorentzian time separation vs. Riemannian distance

Length of causal curves $\gamma : I \to M$: $\quad L_g(\gamma) := \int_I \sqrt{-g(\dot\gamma, \dot\gamma)}$

Null curves have $L_g(\gamma) = 0$, timelike can be arbirarily short!

Time separation function

$\tau(p, q) := \sup\{L_g(\gamma) : \gamma \text{ future causal from } p \text{ to } q\}$

Properties:

- lower semi-continuous,
- satisfies reverse Δ-inequality:

$$\tau(p, q) + \tau(q, r) \leq \tau(p, r)$$

- τ is not a distance function

Why: implements basic causality already of SRT (twin paradox)

Consequences: No cheap entry into the world of metric geometry!

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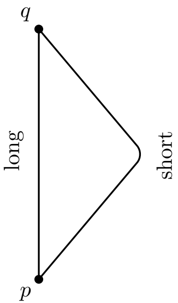

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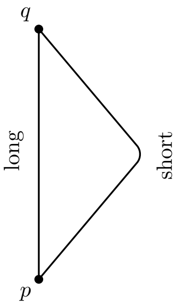

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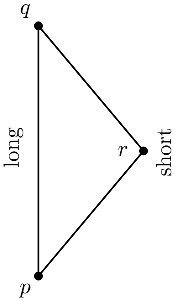

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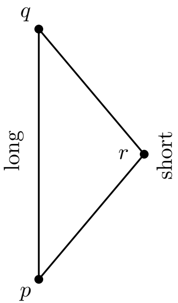

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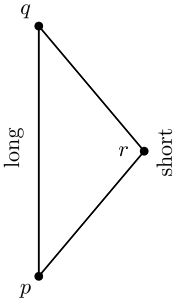

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Causality w/o metric: Lorentzian (pre-)length spaces

Causal space: X (metrizable) topological space with *abstract causality:*

$\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$

Abstract time separation: $\tau\colon X \times X \to [0, \infty]$ lower semicontinuous

Definition [Kunzinger-Sämann, 18]

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Examples
- *smooth spacetimes* (M, g) with usual time separation τ
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Abstract time separation: $\tau\colon X \times X \to [0,\infty]$ lower semicontinuous

Definition [Kunzinger-Sämann, 18]

$(X, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if for $p \leq q \leq r$

$$\tau(p,r) \geq \tau(p,q) + \tau(q,r) \quad \text{and} \quad \tau(p,q) \begin{cases} = 0 & \text{if } x \not\leq y \\ > 0 & \Leftrightarrow x \ll y \end{cases}$$

It is a *Lorentzian length space* if τ is intrinsic.

Examples

- *smooth spacetimes* (M,g) with usual time separation τ
- *Lorentz(-Finsler)* spacetimes, of *low regularity* ($g \in C^0$ +...)
- *finite directed graphs* (causal sets)

Lorentzian *causality theory*

variants by [Braun-McCann, 23-], [Minguzzi-Suhr, 24-], [Müller, 24-]

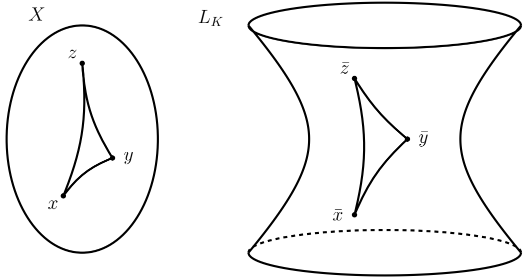
Timelike curvature via triangle comparison

Definition (Synthetic curvature bounds)

$(X, \ll, \leq, \tau)$ has *timelike curvature* $\geq K$ if

1. some technical conditions hold
2. for all *small timelike triangles* Δxyz and their comparison $\Delta\bar{x}\bar{y}\bar{z}$ in L_K and all p, q resp. $\bar{p}$, $\bar{q}$

$$\tau(p,q) \leq \bar{\tau}_K(\bar{p},\bar{q}).$$



Faithful extension of sectional curvature bounds to "metric" Lorentzian setting

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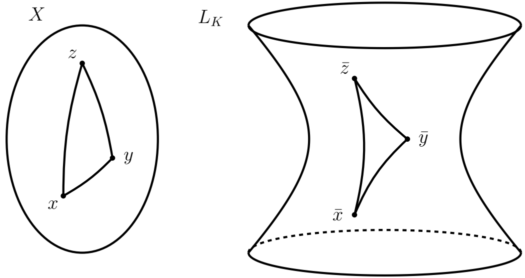

Faithful extension of sectional curvature bounds to "metric" Lorentzian setting

Timelike curvature comparison: The whole truth

Theorem (Equivalence of curvature conditions) [Beran-Kunzinger-Rott, 24]

Let X be a chronological LpLS. Recall curvature bds. below/above by K in the following senses:

1. Timelike triangle comp.
2. One-sided timelike triangle comp.
3. Causal triangle comp.
4. One-sided causal triangle comp.
5. Strict causal triangle comp.
6. One-sided strict causal triangle comp.
7. Monotonicity comp.
8. One-sided monotonicity comp.
9. Angle comp.
10. Hinge comp.
11. Timelike four point condition
12. Angle version of timelike four point condition
13. Causal four point condition
14. Strict causal four-point condition
15. τ-convexity (resp. τ-concavity) condition

If X is strongly causal, regular, and locally D_K-geodesic, and in the case of upper curvature bounds additionally is locally causally closed and in the case of lower curvature bounds satisfies (8), then *all notions of curvature bounds are equivalent.*

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- (ix) Angle comp.

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In general the following relations hold:

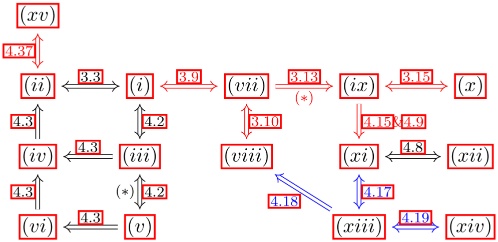

Figure 4: All relations between different formulations of curvature bounds for Lorentzian pre-length space. Black arrows are always valid, red arrows require X to be regular, and blue arrows require X to be strongly causal, regular and locally D_K-geodesic. The two instances of additional assumptions for one direction of curvature bounds are decorated by $(*)$ in the figure.

Selected results (1/3)

Theorem [Kunzinger-Sämann 19, Beran-Sämann, 22]

In a strongly causal Lorentzian pre-length space with *timelike curvature bounded below* timelike geodesics do *not branch*.

Theorem [Grant-Kunzinger-Sämann, 19]

A timelike geodesically complete spacetime (or LLS) is *inextendible as a regular LLS*, i.e., any LLS-extension necessarily has unbounded curvature.

Complements class. [Beem-Ehrlich, 80s], C^0-results [Galloway-Ling-Sbierski, 18] [Minguzzi-Suhr, 19] and relates to *curvature blow up*!

Splitting theorem [Beran-Ohanyan-Rott-Solis, 23]

Let X be a globally hyperbolic LLS with global timelike $K \geq 0$. If X contains a complete timelike line (& some technical conditions) then it splits into a product $\mathbb{R} \times S$ with S a metric length space with $K \geq 0$.

Generalises smooth Lorentzian & synthetic Riemannian results.

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Selected results (2/3): Rauch comparison Theorem

[Grant-Kunzinger-Ohanyan-Schinnerl-S, 25]

- classical Jacobi field estimates
- bound on L of geodesic to 1^{st} conjugate pt.

Classical (Riemannian, C^∞)

$0 < \mathrm{Sec} \leq K \Rightarrow L_h \geq \pi/\sqrt{K}$

Synth. Riem. [Shankar-Sormani, 09]

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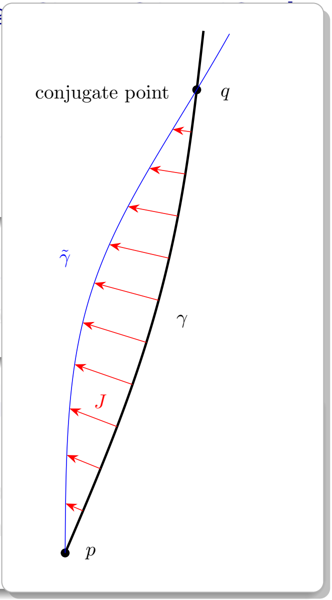

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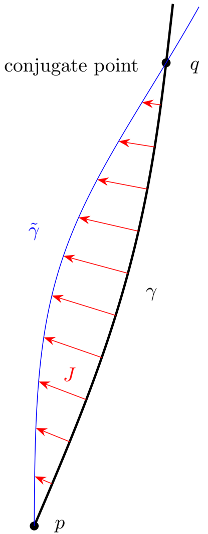

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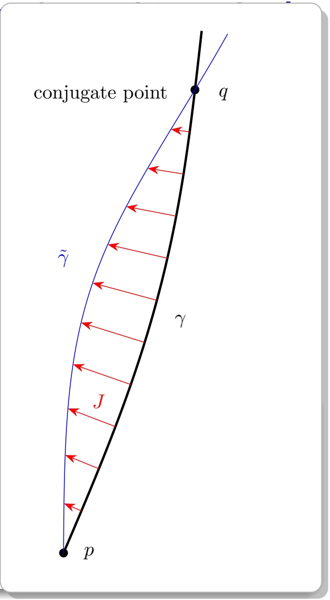

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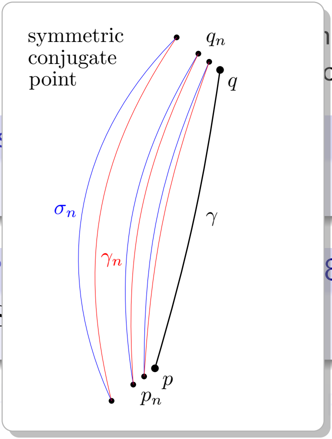

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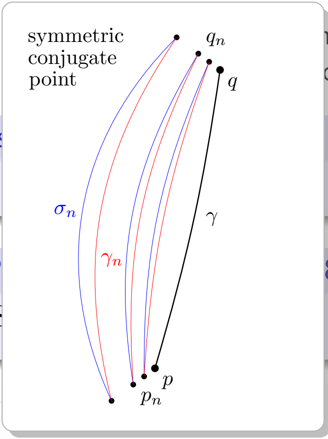

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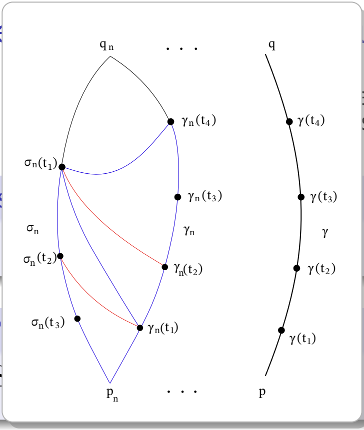

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Closely related to

Synthetic Lorentzian Cartan-Hadamard [Erös-Gieger, 25]

Let X be a globally hyperbolic regular Lorentzian length space. If X is locally concave and future one-connected, then every pair of timelike related points is connected by a unique timelike geodesic, and these geodesics vary continuously with their endpoints.

Selected results (3/3)

- *Time functions* [Burtscher-García-Heveling, 21]
- *Null distance & LLS* [Kunzinger-S., 22]
- compatibility w *distr. curv. bds* [Erös-Kunzinger-Ohanyan-Vardabasso, 25]
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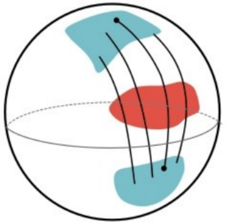

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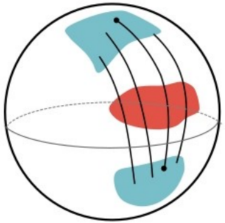
- *Minimize*

$$\int_{X\times Y} c(x,y)\,\mathrm{d}\pi(x,y) \qquad (c\text{...cost function})$$

  over transport plans $\pi \in \mathcal{P}(X \times Y)$
  w given marginals $(\mathrm{pr}_X)_\sharp\pi = \mu_1$, $(\mathrm{pr}_Y)_\sharp\pi = \mu_2$

- What is *optimal* depends on cost,
  but also on *distances* & *geometry*!
- Turn this on its head:
  define *curvature* by requiring that
  OT behaves as in model spaces

- Riemannian case: cost $c = d$
- Lorentzian case: cost $c = \tau$



Transporting *clouds* of points on the sphere

Ricci Bounds via Optimal Transport: Riemannian case

Thm. (Displacement convexity) [Lott-Villani, Sturm 06-09]

(M,h) complete Riemannian manifold

$$\mathrm{Ric} \geq 0 \Longleftrightarrow (M, d_h, \mathrm{vol}_h) \text{ is an } \mathrm{CD}(0,\infty)\text{-space}$$

Definitions. On a metric measure space $(X, d, \mathfrak{m})$ define

- *Wasserstein distance:* $W_2(\mu_1,\mu_2) = \left(\inf_{\pi\in\Pi} \int_{X\times X} d(x,y)^2\, d\pi(x,y)\right)^{\frac{1}{2}}$
- *Wasserstein geodesic:* continuous curve $(\mu_t)_{0\leq t\leq 1}$

$$W_2(\mu_s,\mu_t) = |t-s|\cdot W_2(\mu_1,\mu_2)$$

- *Entropy functional:* $E(\mu|\mathfrak{m}) = \int_X \rho\log\rho\, d\mathfrak{m}$ for $\mu = \rho\mathfrak{m}$
- $\mathrm{CD}(0,\infty)$*-space:* E convex along Wasserstein geodesics, i.e.,

$$E(\mu_t|\mathfrak{m}) \leq (1-t)E(\mu_1|\mathfrak{m}) + tE(\mu_2|\mathfrak{m})$$

Again turn this into definition of synthetic curvature bounds...CD(K,N)
$\rightsquigarrow$ $(R)CD$*-spaces:* stability u. measured GH-conv. [Ambrosio-Gigli-Savare, 14]

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Ricci Bounds via Optimal Transport: Lorentzian case

Thm. (Lor. displacement convexity) [McCann, Mondino-Suhr, 20]

(M, g) globally hyperbolic n-dim. spacetime

$\mathrm{Ric}(X, X) \geq 0$ for X timelike $\Longleftrightarrow$ $(M, \tau, \mathrm{vol}_g)$ is $\mathrm{TCD}(0, n)$-space

Definitions. Measured Lorentzian pre-length space $(X, \mathfrak{m}, \ll, \leq, \tau)$

- OT & causality [Eckstein-Miller, 17]: $\pi \in \Pi_{\leq}$ (concentrated on $\ll$)
- *p-Lorentz Wasserstein distance:* $(0 < p \leq 1)$ (for reverse $\triangle$-ineq.)

$$l_p(\mu_1, \mu_2) = \Big(\sup_{\pi \in \Pi_{\leq}} \int_{X \times X} \tau(x, y)^p \, d\pi(x, y)\Big)^{1/p}$$

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$$e''(t) - 1/N \, e'(t)^2 \geq K \int_{X \times X} \tau(x, y)^2 \pi(dxdx)$$

Turn into definition: measured Lorentzian pre-length space w TCD(K,N)

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ingredients: causal set $(X, \leq)$, partial order
locally finite: $J(x, y) = \{z : x \leq z \leq y\}$ finite

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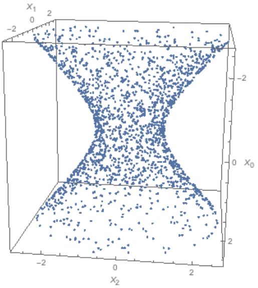

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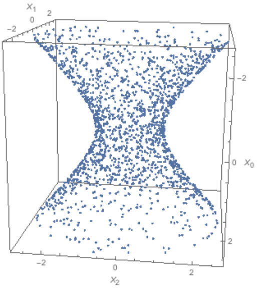

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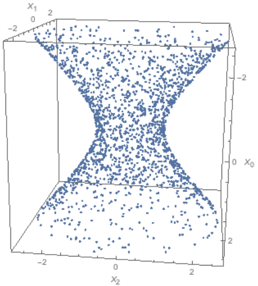

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- chain: $C := (x_i)_{i=1}^n$: $x_i < x_{i+1}$
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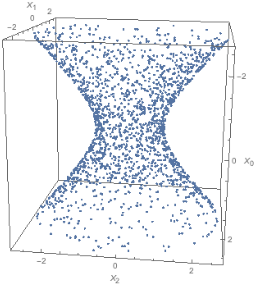

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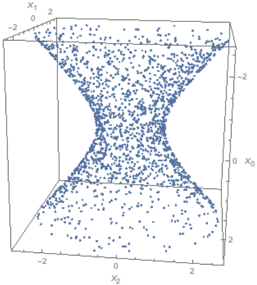

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References

M. Kunzinger, C. Sämann,
Lorentzian length spaces. Ann. Global Anal. Geom. 54, no. 3, 399–447, 2018.

F. Cavalletti, A. Mondino,
Optimal transport in Lorentzian synthetic spaces, synthetic timelike Ricci curvature lower bounds and applications. Cambridge Journal of Mathematics, Camb. J. Math. 12(2) 2024.

M. Kunzinger, A. Ohanyan, B. Schinnerl, R. Steinbauer,
The Hawking-Penrose singularity theorem for C^1-Lorentzian metrics. Commun. Math. Phys. 391, 1143–1179, 2022.

M. Calisti, M. Graf, E. Hafemann, M. Kunzinger, R. Steinbauer,
Hawking's singularity theorem for Lipschitz Lorentzian Metrics. Commun. Math. Phys. 406, 207, 2025.

M. Kunzinger, R. Steinbauer,
Null distance and convergence of Lorentzian length spaces. Ann. Henri Poincare, 23, 2022.

R. Steinbauer,
The singularity theorems of General Relativity and their low regularity extensions. Jahresber. Dtsch. Math.-Ver. 125(2), 2023.

T. Beran, M. Braun, M. Calisti, N. Gigli, R. McCann, A. Ohanyan, F. Rott, C. Sämann,
A nonlinear d'Alembert comparison theorem and causal differential calculus on metric measure spacetimes. arXiv:2408.15968.

M. Braun, N. Gigli, R. McCann, A. Ohanyan, C. Sämann,
An elliptic proof of the splitting theorems from Lorentzian geometry. arXiv:2410.12632.

Meet the EF-Geometry Team

Thank you for your attention

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- slide 3, Cartoon-Einstein: https://wallpapers.com/albert-einstein-png (freely downloadable)
- slide 3, Spacetime-curvature: Einstein relatively easy, https://einsteinrelativelyeasy.com/
- slide 4, lightcone : arXiv:1607.04202[gr-qc], Le Tiec, Alexandre and Novak, Jerome, Theory of Gravitational Waves. arXiv:1607.04202, https://doi.org/10.48550/arXiv.1607.04202
- slide 4, cone-trajectory: Stanford Encyclopedia of Philosophy https://plato.stanford.edu/entries/spacetime-singularities/lightcone.html
- slide 11, spacetime bubble: Chruściel, Piotr T., James, D. E., On Lorentzian causality with continuous metrics, arXiv:1111.0400 [gr-qc], https://doi.org/10.48550/arXiv.1111.0400
- slide 32, spacetime-sprinkling: Sumati Surya, The causal set approach to quantum gravity. Living Reviews in Relativity (2019) 22:5 https://doi.org/10.1007/s41114-019-0023-

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